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Helicity Formalism for Majorana Neutrino Fields in Matter. I. Theoretical Formulation and the MSW Effect

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Abstract

The helicity formalism for the Majorana neutrino fields in matter is presented and used to derive the effective spinorial wave functions for general (n, m) models in which there are n neutrino fields belonging to $SU(2)_L$ isodoublets and m isosinglet neutrino fields. As an application, a rigorous derivation of the MSW time evolution equation is given, with special emphasis on the differences between the mixing of the fields and the states as well as those between the mixing in vacuum and in matter.

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1 Introduction

Among all the particles that are known to exist, the neutrinos are the most elusive and mysterious, and very little is known about their basic properties such as mass, mixing parameters and electromagnetic characteristics. In the Standard Model, which has been extremely successful so far, the neutrinos are massless and interact with other particles only through the weak interactions. On the other hand, it is generally believed that the Standard Model is incomplete and a more satisfactory description of the particle physics phenomena, beyond the Standard Model, has long been sought after. In many of these models the neutrinos are massive Majorana particles. Furthermore, the neutrinos are protagonist of the solar neutrino problem which appears, at this moment, to demand an explanation involving some new physics beyond the Standard Model. Therefore, it is extremely important to study, as rigorously as possible, the effective properties of the neutrinos in matter in order to interpret the experimental data and to constrain the theoretical models.

It has been discovered in the past years that the effective properties of the neutrinos can change drastically while they propagate in a medium. By using a coherent formalism introduced by Wolfenstein [1] to describe the behaviour of neutrinos in matter, Mikheyev and Smirnov [2] demonstrated the possibility of resonant oscillations of the neutrino flavour in matter, known as the MSW effect. The MSW effect can play a crucial role in determining the nature of neutrinos by probing neutrinos from astrophysical objects. In essence, the MSW effect is a consequence of the modification of the energy-momentum dispersion relations for neutrinos propagating in matter by the addition of the effective potentials which describe the coherent interaction of the neutrinos with the particles in the medium.

There are two main issues involved in the MSW effect. The first is *the derivation of the effective potentials* which the neutrinos see when they propagate in matter. There are many papers in which the potentials are derived in different ways, from the original paper by Wolfenstein [1] (see also Ref.[3, 4]) to the finite temperature approach [5, 6].

The second issue, which is more or less taken for granted, is *how to incorporate the potentials in a rigorous way into the equation of motion for the weak eigenstate neutrinos in matter.* The usual practice has been simply to add the potential by hand into the equation of motion in vacuum. There are no theoretical justifications for this, except intuition.

In the phenomenological approach introduced by Wolfenstein, the time evolution equation of the weak states in vacuum in the two generation case

$$\left(i\frac{d}{dt} - P\right)\begin{pmatrix} |\nu_e(t)\rangle \\ |\nu_\mu(t)\rangle \end{pmatrix} = \frac{1}{4P}\left\{\left(m_1^2 + m_2^2\right) + \begin{pmatrix} -\Delta\cos(2\vartheta) & \Delta\sin(2\vartheta) \\ \Delta\sin(2\vartheta) & \Delta\cos(2\vartheta) \end{pmatrix}\right\}\begin{pmatrix} |\nu_e(t)\rangle \\ |\nu_\mu(t)\rangle \end{pmatrix} \tag{1}$$

($\Delta \equiv m_2^2 - m_1^2$, ϑ is the mixing angle in vacuum and P is the magnitude of the three-momentum) is modified in matter with addition "by hand" of the potential energy terms due to the coherent forward elastic scattering with the electrons, protons and neutrons in the medium

$$\left(i\frac{d}{dt}-P-V_N\right)\begin{pmatrix}|\nu_e^{(-)}(\vec{p};t)\rangle\\ |\nu_\mu^{(-)}(\vec{p};t)\rangle\end{pmatrix}=\frac{1}{4P}\left\{\left(m_1^2+m_2^2\right)+2PV_C+\begin{pmatrix}-\Delta\cos\left(2\vartheta\right)+2PV_C & \Delta\sin\left(2\vartheta\right)\\ \Delta\sin\left(2\vartheta\right) & \Delta\cos\left(2\vartheta\right)-2PV_C\end{pmatrix}\right\}\begin{pmatrix}|\nu_e^{(-)}(\vec{p};t)\rangle\\ |\nu_\mu^{(-)}(\vec{p};t)\rangle\end{pmatrix} \tag{2}$$

where $V_C = \sqrt{2}G_F N_e$ and $V_N = -G_F N_n/\sqrt{2}$ are due to weak charged current and neutral current interactions, respectively (N_e and N_n are the electron and neutron number densities in the medium, respectively), and the neutrino state is taken to have negative helicity, denoted by the superscript $^{(-)}$ (Eqs.(1) and (2) are valid in the relativistic approximation). For an appropriate value of the electron number density N_e, depending on the vacuum oscillation parameters Δ and ϑ, the two diagonal elements of the energy matrix in Eq.(2) can be equal, leading to the well known MSW resonance effect, which can explain the depletion of the observed flux of the solar neutrinos in a natural way for a wide range of values of the vacuum oscillation parameters [7]. The energy eigenvalues $E_1^{(-)}$ and $E_2^{(-)}$ and the effective mixing angle $\varphi^{(-)}(P)$ of the negative helicity weak states in matter can be obtained by diagonalizing Eq.(2):

$$\begin{aligned}
E_j^{(-)} &= P + V_N + \frac{1}{4P}\Big[(m_1^2+m_2^2)+2PV_C\Big] \pm \frac{1}{4P}\Delta^{(-)}\\
\Delta^{(-)} &= \sqrt{\Big[\Delta\cos\left(2\vartheta\right)-2PV_C\Big]^2+\Big[\Delta\sin\left(2\vartheta\right)\Big]^2}\\
\tan\Big(2\varphi^{(-)}(P)\Big) &= \frac{\Delta\sin(2\vartheta)}{\Delta\cos(2\vartheta)-2PV_C}
\end{aligned} \tag{3}$$

The vacuum equation (1) and the MSW equation (2) are the phenomenological equations which describe the time evolution of the weak states and the mixing between the weak states and the energy eigenstates. Therefore, it is important to understand the foundations of these equations, i.e. under what approximations they are valid.

In the MSW treatment of neutrino oscillation Eq.(2) is the starting point from which the energy eigenstates, the energy eigenvalues and the mixing angle are derived. This approach has been widely accepted but it lacks a theoretical basis. In particular *the weak states can be defined only as appropriate superpositions of energy eigenstates* that are more fundamental; hence it is not clear what Eq.(2) really means if the energy eigenstates are not defined. In the usual phenomenological MSW approach the energy eigenstates are defined by the diagonalization of Eq.(2), but the correct procedure in a quantum field theoretical approach is to solve the field equations, build the Fock space for the system and derive the behaviour of the states: this is why in our approach Eq.(2) is a final result rather than a starting point, as we shall demonstrate in Section 4.

Moreover, the MSW equation cannot say anything about the neutrino fields in matter, whose structure can be found only by solving the field equation. In vacuum the weak fields are mixed due to the off-diagonal mass terms in the Lagrangian, but it is quite clear that the effect of matter is not a simple modification in the mixing of the fields since the mixing angle in matter, as given by Eq.(3), is momentum dependent and a field contains a superposition of components corresponding to all the momenta ranging from zero to infinity. To our knowledge only Mannheim [10] has clearly emphasized this fact, while many authors do not make any distinction between states and fields. Although this ambiguity is often academic, without practical consequences, sometimes it can lead to confusion. This is the case of a paper [11] in which the authors calculate the effective majoron decay rate in matter by mixing the neutrino fields in order to obtain off-diagonal elements in the majoron coupling matrix.

Several papers have been published in which the MSW time evolution equation of the neutrino oscillation in matter is derived in the framework of a field theoretical treatment of the neutrino field. One method, by Halprin [8], is to eliminate the spin structure in the Dirac equation by reducing it to a Klein-Gordon type equation (see also Ref.[9]). This approach, although elegant for deriving the MSW equation, does not provide any information about the effective spinorial wave functions of the neutrinos in matter. The knowledge of the effective neutrino fields in matter in the presence of flavour mixing is essential in order to perform realistic (and reliable) calculations of neutrino decays (such as radiative or majoron decays) or cross sections in matter. An appropriate formalism has been developed by Mannheim [10] by using the two-component Dirac and Majorana neutrino fields in order to solve the coupled field equations in matter in the two generation case.

In this paper, which is the first of two, we follow the approach of Mannheim and derive explicitly the effective spinorial wave functions of neutrinos in matter for general (n, m) models in which there are n neutrino fields belonging to $SU(2)_L$ isodoublets and m isosinglet neutrino fields. We have chosen this framework since, in general, neutrinos can have both Majorana and Dirac mass terms and the existence of isosinglet neutrino fields provides a natural explanation of the smallness of the light neutrino masses through the see-saw mechanism. The formalism can be extended in a straightforward way to the case in which right handed weak currents exist and to any massive Majorana particles as well as to Dirac neutrinos. As an application, we show in Section 4 that the MSW time evolution equation for the two generation case can be derived in a very transparent way. In the second paper [12] (referred to as paper II) we apply the formalism to study the majoron decays of neutrinos propagating in matter. We show that the knowledge of the effective spinorial wave functions is necessary in order to calculate the neutrino decays in matter.

In Section 2 we present the canonical quantization of a Majorana neutrino field in matter, in the two-component formalism, following the approach of Mannheim [13]. In Section 3 we generalize the formalism to general (n, m) models in which there are n neutrino fields belonging to $SU(2)_L$ isodoublets and m isosinglet neutrino fields. In Section 4 we discuss the problem of defining the weak neutrino states in vacuum and in matter, in order to obtain the MSW equation (2) as an application of the formalism developed in Section 3.

Throughout this paper the greek indices $\alpha, \beta, \ldots$ refer to the neutrino fields in the weak basis (and related quantities), the first latin indices $a, b, \ldots$ refer to the neutrino fields in the mass basis (and related quantities) and the middle latin indices $j, k, \ldots$ refer to the energy eigenstate neutrinos propagating in matter.

2 Majorana Neutrino Field in Matter

When a neutrino propagates in matter, its energy is changed by the coherent weak interactions with the particles in the medium. The coherence requirement constrains the medium to remain unchanged in order to allow the interference of the scattered and unscattered neutrino wave functions. It should be emphasized that the initial and final states of the scatterers in the medium must be exactly the same since any change to orthogonal states, regardless of whether an energy change is involved or not, will produce incoherent waves that do not interfere with the unscattered wave. The coherence requirement allows one to write the weak interaction Lagrangian only in terms of the neutrino field and the medium effect is described by an optical potential that depends on the matter density and composition. Therefore the neutrino field equation is decoupled from the many-body field equations of the medium and can be solved. The effective weak interaction Lagrangians that describe the medium effect are

$$
\begin{aligned}
&\mathcal{L}_W^{(\alpha)}(x) = -V_\alpha\ \Phi_\alpha^\dagger(x)\Phi_\alpha(x) \qquad\qquad \alpha = e, \mu, \tau \\
&V_e = V_C + V_N = \frac{G_F}{\sqrt{2}}\,[2N_e - N_n] \\
&V_\mu = V_\tau = V_N = -\frac{G_F}{\sqrt{2}}\,N_n
\end{aligned}
\tag{4}
$$

where $\Phi_\alpha(x)$ are two-component spinor fields that describe Majorana neutrinos in the weak interaction basis. Although the values of V_C and V_N are the same as the values of the usual potential energies denoted by the same notation, here V_C and V_N are optical potentials in the effective Lagrangian that describe the effect of the weak charged current and neutral current interactions with the particles in the medium, respectively. In order to work out the potential energy for a propagating electron neutrino, one must calculate the effective forward elastic scattering amplitudes in matter, which give the usual result in the relativistic approximation $(m \ll P)$

$$
\mathcal{A}(\nu_\alpha(p,\pm) \to \nu_\alpha(p,\pm)) = \langle \nu_\alpha(p,\pm)| - \int d\vec{x}\ \mathcal{L}_W^{(\alpha)}(x)|\nu_\alpha(p,\pm)\rangle \simeq \mp V_\alpha \tag{5}
$$

Let us consider a generic Majorana neutrino described by the two-component spinor field $\Phi(x)$, whose optical potential in matter is V. The effective Lagrangian in matter is given by

$$
\mathcal{L}_{\text{eff}}(x) = \Phi^\dagger(x)\left[i\partial_0 - i\vec{\sigma}\cdot\vec{\nabla}\right]\Phi(x) - \frac{m}{2}\left[\Phi^T(x)i\sigma^2\Phi(x) - \Phi^\dagger(x)i\sigma^2\Phi^*(x)\right] - V\Phi^\dagger(x)\Phi(x) \tag{6}
$$

where the superscript * is used to denote hermitian conjugation in the operator space and complex conjugation in the spinor space. The simple form of the potential term in the effective Lagrangian (6) clearly explains why we choose to use the two-component formalism in which we do not have to deal with Dirac matrices; this advantage will be essential in solving the two generation case. In our approach we assume that the optical potential V is spatially constant; this is a good approximation as long as the variation of the potential within a neutrino wavelength is negligible in comparison with the values of the parameters having dimension of energy (energy, momentum, mass and the potential itself). The field equation is

$$\left[i\partial_0 - i\vec{\sigma}\cdot\vec{\nabla} - V\right]\Phi(x) + mi\sigma^2\Phi^*(x) = 0 \tag{7}$$

Since the helicity is conserved by the weak interactions of the neutrino with the medium, we solve the field equation (7) by expanding the field $\Phi(x)$ as a superposition of plane wave spinors with definite helicity

$$\Phi(x) = \int \frac{d\vec{p}}{\sqrt{(2\pi)^3}}\, \mathrm{e}^{i\vec{p}\cdot\vec{x}} \sum_{h=\pm1} \left[\mathcal{A}^{(h)}(\vec{p},t) + \mathcal{B}^{(h)}(\vec{p},t)\right] w(\vec{p},h) \tag{8}$$

where $h=\pm1$ denote the helicity and $w(\vec{p},\pm)$ are two-component ortho-normalized helicity eigenstate spinors which satisfy the following equations [14]

$$\begin{aligned}
&\vec{\sigma}\cdot\vec{p}\ w(\vec{p},h) = h\ P\ w(\vec{p},h) \\
&i\sigma^2\ w^*(\vec{p},h) = -h\ w(\vec{p},-h) \\
&w(-\vec{p},-h) = \eta(\vec{p},h)\ w(\vec{p},h) \\
&\eta^*(\vec{p},h) = -\eta(\vec{p},-h) \\
&\eta(\vec{p},h) = -\eta(-\vec{p},h)
\end{aligned} \tag{9}$$

where $P \equiv |\vec{p}|$ and $\eta(\vec{p},\pm)$ are phase factors. In Eq.(8) $\mathcal{A}^{(h)}(\vec{p},t)$ and $\mathcal{B}^{(h)}(\vec{p},t)$ denote the positive and negative frequency parts of the field, respectively,

$$\begin{aligned}
&\mathcal{A}^{(h)}(\vec{p},t) = \mathcal{A}^{(h)}(\vec{p})\ \mathrm{e}^{-iE^{(h)}t} \\
&\mathcal{B}^{(h)}(\vec{p},t) = \mathcal{B}^{(h)}(\vec{p})\ \mathrm{e}^{iE^{(h)}t}
\end{aligned} \tag{10}$$

where we have made a distinction between $E^{(+)}$ and $E^{(-)}$ because the positive and negative helicity states interact in a different way with matter and hence have different energies. The field equation (7) yields two equations corresponding to the two helicity eigenvalues $h=\pm1$

$$\begin{pmatrix} E^{(h)} + h\,P - V & -m \\ -m & E^{(h)} - h\,P + V \end{pmatrix} \begin{pmatrix} \mathcal{A}^{(h)}(\vec{p}) \\ h\ \eta(\vec{p},h)\ \mathcal{B}^{(h)*}(-\vec{p}) \end{pmatrix} = 0 \tag{11}$$

The determinantal condition associated with Eq.(11) gives the dispersion relations

$$E^{(h)2} = (h\,P - V)^2 + m^2 \tag{12}$$

The dispersion relation is different for the two helicity eigenvalues due to the interactions with the medium, described by the effective potential V.

In order to quantize properly the field $\Phi(x)$, we define

$$\begin{aligned} \mathcal{A}^{(h)}(\vec{p}) &= \sqrt{\frac{E^{(h)} - h\,P + V}{2E^{(h)}}}\; a(\vec{p},h) \\ h\;\eta(\vec{p},h)\;\mathcal{B}^{(h)*}(-\vec{p}) &= \sqrt{\frac{E^{(h)} + h\,P - V}{2E^{(h)}}}\; a(\vec{p},h) \end{aligned} \tag{13}$$

where $a(\vec{p},\pm)$ are operators that obey the canonical anticommutation relations. In this way we obtain the correct normal ordered Hamiltonian in order to build the Fock space for the field $\Phi(x)$ in matter

$$:\mathcal{H}(x): = \int d\vec{p}\;\left\{E^{(-)}a^{\dagger}(\vec{p},-)a(\vec{p},-) + E^{(+)}a^{\dagger}(\vec{p},+)a(\vec{p},+)\right\} \tag{14}$$

One can now interpret $a(\vec{p},\pm)$ and $a^{\dagger}(\vec{p},\pm)$ as the destruction and creation operators of the one particle neutrino states in matter. The difference between the states in vacuum and in matter lies in the dispersion relation between energy and momentum; in matter the dispersion relations for the two helicity eigenstates are given by Eq.(12).

The plane wave expansion of the quantized two-component Majorana field $\Phi(x)$ in matter is

$$\begin{aligned} \Phi(x) = \int \frac{d\vec{p}}{\sqrt{(2\pi)^3}} \sum_{h=\pm1} \Bigg\{ & \sqrt{\frac{E^{(h)} - h\,P + V}{2E^{(h)}}}\; w(\vec{p},h)\; a(\vec{p},h)\; \mathrm{e}^{-iE^{(h)}t + i\vec{p}\cdot\vec{x}} + \\ & -h\;\sqrt{\frac{E^{(h)} + h\,P - V}{2E^{(h)}}}\; w(\vec{p},-h)\; a^{\dagger}(\vec{p},h)\; \mathrm{e}^{iE^{(h)}t - i\vec{p}\cdot\vec{x}} \Bigg\} \end{aligned} \tag{15}$$

The formalism adopted for a Majorana neutrino field can be easily generalized to a Dirac field: we leave it as an exercise for the reader. Here we give only the result in the relativistic approximation (quite obvious for the experts): The results obtained for the negative helicity sector are the same for a Dirac or Majorana neutrino, whereas the behaviour of a right-handed Dirac anti-neutrino is the same as the behaviour of a right-handed Majorana neutrino, justifying the usual practice to call it antineutrino (of course in the relativistic approximation right handed Dirac neutrinos and left handed Dirac anti-neutrinos do not interact with matter).

3 Mixing in Matter

In this Section we generalize the formalism presented in the previous Section to general (n,m) models in which there are n neutrino fields belonging to $SU(2)_L$ isodoublets and m isosinglet neutrino fields. We will denote the two-component neutrino fields in the weak basis by $\Phi_{W\alpha}(x)$ where $\alpha = 1,\ldots,n$ for the n neutrino fields belonging to $SU(2)_L$ isodoublets and

$\alpha = n+1, \ldots, N$ for the m isosinglet neutrino fields, with $N \equiv n+m$. The Lagrangian in vacuum is

$$\mathcal{L}(x) = \Phi_W^\dagger(x) \left[i\partial_0 - i\vec{\sigma}\cdot\vec{\nabla} \right] \Phi_W(x) - \frac{1}{2} \left[\Phi_W^T(x) M_W i\sigma^2 \Phi_W(x) - \Phi_W^\dagger(x) M_W^\dagger i\sigma^2 \Phi_W^*(x) \right] \quad (16)$$

where M_W is a $N \times N$ complex symmetric mass matrix that can be decomposed as

$$M_W = \begin{pmatrix} M_W^{(n)} & M_W^{(D)} \\ M_W^{(D)T} & M_W^{(m)} \end{pmatrix} \quad (17)$$

where $M_W^{(n)}$ and $M_W^{(m)}$ are two complex symmetric $n \times n$ and $m \times m$ Majorana mass matrices, respectively, and $M_W^{(D)}$ is a complex $n \times m$ Dirac mass matrix. The mass matrix M_W can be diagonalized by a unitary transformation

$$\mathcal{U}^T M_W \mathcal{U} = M_M \quad (18)$$

where $\mathcal{U}$ is a $N \times N$ unitary matrix ($\mathcal{U}^\dagger \mathcal{U} = \mathcal{U}\mathcal{U}^\dagger = 1$) and M_M is a $N \times N$ diagonal matrix with real and positive definite eigenvalues ($(M_M)_{ab} = m_a \delta_{ab}$, $m_a = m_a^* \geq 0$) [15, 16]. The two-component mass eigenstate neutrino fields $\Phi_{Ma}(x)$ ($a = 1, \ldots, N$) defined by

$$\Phi_{W\alpha}(x) = \sum_{a=1}^{N} \mathcal{U}_{\alpha a} \Phi_{Ma}(x) \quad (19)$$

diagonalize the vacuum Lagrangian

$$\mathcal{L}(x) = \sum_{a=1}^{N} \left\{ \Phi_{Ma}^\dagger(x) \left[i\partial_0 - i\vec{\sigma}\cdot\vec{\nabla} \right] \Phi_{Ma}(x) - \frac{m_a}{2} \left[\Phi_{Ma}^T(x) i\sigma^2 \Phi_{Ma}(x) - \Phi_{Ma}^\dagger(x) i\sigma^2 \Phi_{Ma}^*(x) \right] \right\} \quad (20)$$

The charged lepton current in the weak interaction Lagrangian is given by

$$\begin{aligned} j^\mu(x) &= 2 \sum_{\alpha=1}^{n} \overline{\ell_{\alpha L}}(x)\ \gamma^\mu\ \nu_{W\alpha L}(x) \\ &= 2 \sum_{\alpha=1}^{n} \sum_{a=1}^{N} \overline{\ell_{\alpha L}}(x)\ \gamma^\mu\ \mathcal{K}_{\alpha a}\ \nu_{MaL}(x) \end{aligned} \quad (21)$$

where $\ell_{\alpha L}(x)$ ($\alpha = 1, \ldots, n$) are the left-handed charged lepton fields, the left-handed four-component neutrino fields $\nu_{MaL}(x)$ and the two-component neutrino fields $\Phi_{MaL}(x)$ are related by Eq.(17) (an analogous relation holds in the weak basis) and we have defined for convenience the $n \times N$ rectangular matrix $\mathcal{K}$ such that $\mathcal{K}_{\alpha a} = \mathcal{U}_{\alpha a}$ for $\alpha = 1, \ldots, n$ and $a = 1, \ldots, N$. The rectangular matrix $\mathcal{K}$ is the analog of the Cabibbo-Kobayashi-Maskawa mixing matrix for the present case. It satisfies $\mathcal{K}\mathcal{K}^\dagger = 1$, but $\mathcal{K}^\dagger\mathcal{K} \neq 1$ for $m \neq 0$. In general, the mixing matrix $\mathcal{K}$ contains $n(n+2m-1)$ real physical parameters of which half are mixing angles and half are CP violating phases [15].

The effective weak interaction Lagrangian in matter in the weak basis is

$$\mathcal{L}_I(x) = -\sum_{\alpha=1}^{n} \Phi^\dagger_{W\alpha}(x) V_{W\alpha\alpha} \Phi_{W\alpha}(x) \tag{22}$$

where the potential matrix V_W is diagonal, with entries $V_{W11} = V_N + V_C$ and $V_{W\alpha\alpha} = V_N$ for $\alpha = 2, \ldots, n$ in an electron rich medium.

The effective Lagrangian in matter in the mass basis is

$$\begin{aligned}\mathcal{L}_{\text{eff}}(x) = \sum_{a=1}^{N} \Big\{ \Phi^\dagger_{Ma}(x) \left[i\partial_0 - i\vec{\sigma}\cdot\vec{\nabla} \right] \Phi_{Ma}(x) - \frac{m_a}{2} \left[\Phi^T_{Ma}(x) i\sigma^2 \Phi_{Ma}(x) - \Phi^\dagger_{Ma}(x) i\sigma^2 \Phi^*_{Ma}(x) \right] \Big\} \\ - \sum_{a,b=1}^{N} \Phi^\dagger_{Ma}(x) V_{Mab} \Phi_{Mb}(x)\end{aligned} \tag{23}$$

where

$$V_{Mab} = \sum_{\alpha=1}^{n} \mathcal{K}^*_{\alpha a} V_{W\alpha\alpha} \mathcal{K}_{\alpha b} \tag{24}$$

The mass basis diagonalizes the vacuum Hamiltonian and it is the correct one to use in order to quantize the neutrino fields in vacuum. The weak fields $\Phi_{W\alpha}(x)$ in vacuum are a superposition of the quantized mass eigenstate field $\Phi_{Ma}(x)$, with the mixing matrix $\mathcal{U}$. From the form of the effective Lagrangian in matter given in Eq.(23), it is quite clear that there is no further mixing of the fields in matter since it is not possible to mix the fields in order to diagonalize simultaneously the mass and potential terms. Nevertheless it is possible to solve the field equations by following a procedure analogous to that employed in the previous Section.

The N coupled field equations in the mass basis are given by

$$\left[i\partial_0 - i\vec{\sigma}\cdot\vec{\nabla} \right] \Phi_{Ma}(x) + m_a i\sigma^2 \Phi^*_{Ma}(x) - \sum_{b=1}^{N} V_{Mab} \Phi_{Mb}(x) = 0 \qquad a = 1, \ldots, N \tag{25}$$

We solve the field equations by expanding the fields $\Phi_{Ma}(x)$ as superpositions of plane wave spinors with definite helicity

$$\Phi_{Ma}(x) = \int \frac{d\vec{p}}{\sqrt{(2\pi)^3}}\, \mathrm{e}^{i\vec{p}\cdot\vec{x}} \sum_{h=\pm1} \left[\mathcal{A}^{(h)}_a(\vec{p},t) + \mathcal{B}^{(h)}_a(\vec{p},t) \right] w(\vec{p},h) \tag{26}$$

where $h = \pm1$ denote the helicity, $\mathcal{A}^{(h)}_a(\vec{p},t)$ and $\mathcal{B}^{(h)}_a(\vec{p},t)$ denote the positive and negative frequency parts of the fields, respectively, and $w(\vec{p},h)$ are two-component ortho-normalized helicity eigenstate spinors which satisfy the equations (9). In order to quantize properly the fields $\Phi_{Ma}(x)$, we expand the positive and negative frequency parts $\mathcal{A}^{(h)}_a(\vec{p},t)$ and $\mathcal{B}^{(h)}_a(\vec{p},t)$ as superpositions of energy eigenstates

$$\begin{aligned}\mathcal{A}^{(h)}_a(\vec{p},t) &= \sum_{j=1}^{N} \alpha^{(h)}_{aj}(P)\; a_j(\vec{p},h)\; \mathrm{e}^{-iE^{(h)}_j t} \\ h\;\eta(\vec{p},h)\; \mathcal{B}^{(h)*}_a(-\vec{p},t) &= \sum_{j=1}^{N} \beta^{(h)}_{aj}(P)\; a_j(\vec{p},h)\; \mathrm{e}^{-iE^{(h)}_j t}\end{aligned} \tag{27}$$

where $P \equiv |\vec{p}|$, $E_j^{(h)}$ ($j = 1, \ldots, N$) are the energy eigenvalues (the sum over j goes from 1 to N because there are N degrees of freedom for each helicity eigenvalue) and $a_j(\vec{p}, h)$ ($j = 1, \ldots, N$) are operators that obey the canonical anticommutation relations. The values of the coefficients $\alpha_{aj}^{(h)}(P)$ and $\beta_{aj}^{(h)}(P)$ for each helicity eigenvalue $h = \pm 1$ are given by the $2N$ coupled equations

$$\left.\begin{array}{l} \left[E_j^{(h)} + h\,P\right] \alpha_{aj}^{(h)}(P) - m_a\,\beta_{aj}^{(h)}(P) - \sum\limits_{b=1}^{N} V_{Mab}\,\alpha_{bj}^{(h)}(P) = 0 \\ \left[E_j^{(h)} - h\,P\right] \beta_{aj}^{(h)}(P) - m_a\,\alpha_{aj}^{(h)}(P) + \sum\limits_{b=1}^{N} V_{Mab}^{*}\,\beta_{bj}^{(h)}(P) = 0 \end{array}\right\} a = 1, \ldots, N \tag{28}$$

Since the system of equations (28) is homogeneous, there are N solutions for $j = 1, \ldots, N$ corresponding to the N energy eigenvalues $E_j^{(h)}$. Since the solutions are orthogonal

$$\sum_{a=1}^{N} \left[\alpha_{ai}^{(h)*}(P)\,\alpha_{aj}^{(h)}(P) + \beta_{ai}^{(h)*}(P)\,\beta_{aj}^{(h)}(P)\right] = \delta_{ij} \tag{29}$$

the interpretation of the operators $a_j(\vec{p}, h)$ and $a_j^{\dagger}(\vec{p}, h)$ as destruction and creation operators obeying the canonical anticommutation rules leads to the correct normal ordered Hamiltonian

$$:\mathcal{H}(x): = \int d\vec{p} \sum_{j=1}^{N} \left\{ E_j^{(-)}\; a_j^{\dagger}(\vec{p}, -)\; a_j(\vec{p}, -) + E_j^{(+)}\; a_j^{\dagger}(\vec{p}, +)\; a_j(\vec{p}, +) \right\} \tag{30}$$

The plane wave expansion in matter of the quantized two component neutrino fields $\Phi_{Ma}(x)$ in the mass basis can be written as

$$\begin{aligned} \Phi_{Ma}(x) = \int \frac{d\vec{p}}{\sqrt{(2\pi)^3}} \sum_{j=1}^{N} \sum_{h=\pm 1} \Big\{ & \alpha_{aj}^{(h)}(P)\; w(\vec{p}, h)\; a_j(\vec{p}, h)\; \mathrm{e}^{-iE_j^{(h)}t + i\vec{p}\cdot\vec{x}} + \\ & -h\; \beta_{aj}^{(h)*}(P)\; w(\vec{p}, -h)\; a_j^{\dagger}(\vec{p}, h)\; \mathrm{e}^{iE_j^{(h)}t - i\vec{p}\cdot\vec{x}} \Big\} \end{aligned} \tag{31}$$

For the calculation of weak interaction processes, it is useful to write the neutrino fields in the usual four component formalism. In the chiral representation of the γ matrices the plane wave expansion in matter of the four component Majorana neutrino fields $\nu_{Ma}(x)$ is

$$\begin{aligned} \nu_{Ma}(x) = \int \frac{d\vec{p}}{\sqrt{(2\pi)^3}} \sum_{j=1}^{N} \sum_{h=\pm 1} \Bigg\{ & \begin{pmatrix} \alpha_{aj}^{(h)}(P) \\ \beta_{aj}^{(h)}(P) \end{pmatrix} w(\vec{p}, h)\; a_j(\vec{p}, h)\; \mathrm{e}^{-iE_j^{(h)}t + i\vec{p}\cdot\vec{x}} + \\ & + \begin{pmatrix} -h\; \beta_{aj}^{(h)*}(P) \\ h\; \alpha_{aj}^{(h)*}(P) \end{pmatrix} w(\vec{p}, -h)\; a_j^{\dagger}(\vec{p}, h)\; \mathrm{e}^{iE_j^{(h)}t - i\vec{p}\cdot\vec{x}} \Bigg\} \end{aligned} \tag{32}$$

For a relativistic neutrino, for which $m_a \ll P$ $(a = 1, \dots, N)$ and $V \ll P$, the energy-momentum dispersion relations

$$E_j^{(\pm)2} = P^2 + m_j^{(\pm)2} \qquad j = 1, \dots, N \tag{33}$$

can be approximated to

$$E_j^{(\pm)} = P + \frac{m_j^{(\pm)2}}{2P} \qquad j = 1, \dots, N \tag{34}$$

where $m_j^{(\pm)2}$ $(j = 1, \dots, N)$ are the effective masses squared in matter of the light neutrinos. Since to lowest order

$$\left.\begin{aligned} \beta_{aj}^{(-)}(P) &= \frac{m_a}{2P}\, \alpha_{aj}^{(-)}(P) \\ \alpha_{aj}^{(+)}(P) &= \frac{m_a}{2P}\, \beta_{aj}^{(+)}(P) \end{aligned}\right\} a = 1, \dots, N \tag{35}$$

the system of equations (28) reduces to

$$\frac{m_j^{(-)} - m_a^2}{2P}\, \alpha_{aj}^{(-)}(P) - \sum_{b=1}^{N} V_{Mab}\, \alpha_{bj}^{(-)}(P) = 0 \qquad a = 1, \dots, N \tag{36}$$

for the negative helicity sector and

$$\frac{m_j^{(+)} - m_a^2}{2P}\, \beta_{aj}^{(+)}(P) + \sum_{b=1}^{N} V_{Mab}^{*}\, \beta_{bj}^{(+)}(P) = 0 \qquad a = 1, \dots, N \tag{37}$$

for the positive helicity sector. The N orthogonal solutions $\alpha_{aj}^{(-)}(P)$ and $\beta_{aj}^{(+)}(P)$ $(a = 1, \dots, N)$ for $j = 1, \dots, N$ are the columns of the $N \times N$ unitary matrices $\alpha^{(-)}(P)$ and $\beta^{(+)}(P)$ that diagonalize the equations

$$\alpha^{(-)\dagger}(P) \left[M_M^2 + 2PV_M \right] \alpha^{(-)}(P) = M^{(-)2}(P) \tag{38}$$

and

$$\beta^{(+)\dagger}(P) \left[M_M^2 - 2PV_M^{*} \right] \beta^{(+)}(P) = M^{(+)2}(P) \tag{39}$$

where $M^{(\pm)2}(P)$ are the diagonal $N \times N$ matrices containing the effective masses squared $m_j^{(\pm)2}(P)$ for $j = 1, \dots, N$. Therefore, in the relativistic approximation, the components of the plane wave expansion of the neutrino fields given in Eq.(31) mix in a simple way through the unitary mixing matrices $\alpha^{(-)}(P)$ and $\beta^{(+)*}(P)$. This leads to the mixing of the states, as will be explained in Section 5. Of course, since the mixing depends on the momentum and a field contains a superposition of an infinite range of momentum components, there is no simple mixing of the fields in matter.

In the spirit of the see-saw mechanism, one can assume that the entries in the mass matrix $M_W^{(m)}$ are very large, of the order of a large mass scale related to a new physics beyond the standard model; in a hypothetical case of three generation where the order of magnitude of the entries in the Dirac mass matrix $M_W^{(D)}$ is taken to be about 1 GeV (tau mass) and the light neutrino mass scale is less than 1 keV, one finds $\mathrm{O}\left(M_W^{(m)}\right) \gtrsim 10^6\,\mathrm{GeV}$. This example leads to large values for the m vacuum mass eigenvalues $m_a \sim \mathrm{O}\left(M_W^{(m)}\right)$ with $a = n+1, \dots, N$ and to very small mixing angles between the triplet and singlet sector, of the order $\mathrm{O}\left(M_W^{(D)}\right)/\mathrm{O}\left(M_W^{(m)}\right)$. For neutrinos with energy of the order 1-10 MeV, for which $P \ll \mathrm{O}\left(M_W^{(m)}\right)$, the heavy sector with $a = n+1, \dots, N$ decouples from the light sector with $a = 1, \dots, n$ (for matter densities $\rho \lesssim 10^{14}\,\mathrm{g/cm^3}$ the order of magnitude of the potentials is $V \lesssim 1\,\mathrm{eV}$). In this case one can approximate the system in Eq.(28) by taking only the $2n$ equations for $a, b = 1, \dots, n$.

As a simple application, let us consider the two generation case $(n, m) = (2, 0)$. The mixing matrices $\mathcal{K}$ and $\mathcal{U}$ are equal and can be written as

$$\mathcal{K} = \mathcal{U} = \begin{pmatrix} \cos\vartheta & \sin\vartheta\ \mathrm{e}^{i\delta} \\ -\sin\vartheta & \cos\vartheta\ \mathrm{e}^{i\delta} \end{pmatrix} \tag{40}$$

where ϑ is the mixing angle and δ is the CP violating phase characteristic of two generations of Majorana neutrinos. Since $V_{W11} = V_N + V_C$ and $V_{W22} = V_N$, the potential matrix in the mass basis is

$$V_M = \mathcal{K}^\dagger V_W \mathcal{K} = \begin{pmatrix} V_N + V_C\cos^2\vartheta & V_C\cos\vartheta\sin\vartheta\ \mathrm{e}^{i\delta} \\ V_C\cos\vartheta\sin\vartheta\ \mathrm{e}^{-i\delta} & V_N + V_C\sin^2\vartheta \end{pmatrix} \tag{41}$$

The fact that the potential matrix depends on the mixing angle ϑ and on the CP violating phase δ is obviously due to the fact that the weak interaction Lagrangian is not diagonal in the mass basis and the mixing depends on the two parameters ϑ and δ, namely the interactions with the electron background can in principle break the CP invariance of the neutrino Lagrangian. The system of $2N = 4$ equations (28) can be written as

$$\begin{pmatrix} \mp P + V_N + V_C\cos^2\vartheta & m_1 & V_C\cos\vartheta\sin\vartheta\mathrm{e}^{i\delta} & 0 \\ m_1 & \pm P - V_N - V_C\cos^2\vartheta & 0 & -V_C\cos\vartheta\sin\vartheta\mathrm{e}^{-i\delta} \\ V_C\cos\vartheta\sin\vartheta\mathrm{e}^{-i\delta} & 0 & \mp P + V_N + V_C\sin^2\vartheta & m_2 \\ 0 & -V_C\cos\vartheta\sin\vartheta\mathrm{e}^{i\delta} & m_2 & \pm P - V_N - V_C\sin^2\vartheta \end{pmatrix} \begin{pmatrix} \alpha_{1j}^{(\pm)}(P) \\ \beta_{1j}^{(\pm)}(P) \\ \alpha_{2j}^{(\pm)}(P) \\ \beta_{2j}^{(\pm)}(P) \end{pmatrix} = E_j^{(\pm)} \begin{pmatrix} \alpha_{1j}^{(\pm)}(P) \\ \beta_{1j}^{(\pm)}(P) \\ \alpha_{2j}^{(\pm)}(P) \\ \beta_{2j}^{(\pm)}(P) \end{pmatrix} \tag{42}$$

This equation (for the negative helicity sector with $\delta = 0$) has previously been derived by

Mannheim in Ref.[10]. The energy eigenvalues are

$$
\begin{aligned}
E_1^{(\pm)2}(P) &= P \mp P(2V_N + V_C) + \frac{1}{2}\Big[m_1^2 + m_2^2 + V_N^2 + (V_N + V_C)^2\Big] - \frac{1}{2}\Delta^{(\pm)} \\
E_2^{(\pm)2}(P) &= P \mp P(2V_N + V_C) + \frac{1}{2}\Big[m_1^2 + m_2^2 + V_N^2 + (V_N + V_C)^2\Big] + \frac{1}{2}\Delta^{(\pm)} \\
\Delta^{(\pm)2} &= \Big[\Delta\cos(2\vartheta) - 2V_C(\mp P + V_N)\Big]^2 + \Big[\Delta\sin(2\vartheta)\Big]^2 + \\
&\quad + V_C^2\Big[V_C^2 + 4V_C(\mp P + V_N) + (m_2 - m_1)^2\sin^2(2\vartheta) - 2\Delta\cos(2\vartheta) + \\
&\quad + 2m_1 m_2 \sin^2(2\vartheta)\Big(1 - \cos(2\delta)\Big)\Big]
\end{aligned}
\tag{43}
$$

where $\Delta \equiv m_2^2 - m_1^2$. For a relativistic left-handed neutrino in ordinary matter ($m_a^2 \ll P$ and $V \ll P$), the approximation of Eq.(43) yields the same values of the energy eigenvalues as those obtained from the MSW equation (2), given in Eq.(3). Note that the exact energy eigenvalues, given in Eq.(43), depend on the CP violating phase δ, but the dependence is extremely weak for relativistic neutrinos, being suppressed by a factor of the order m_a/P in comparison with the leading term given in Eq.(3).

For relativistic neutrinos, the 2×2 mixing matrices $\alpha^{(-)}(P)$ and $\beta^{(+)}(P)$ are parametrized in term of the mixing angles in matter $\vartheta^{(\pm)}(P)$ and the CP violating phase δ as

$$
\alpha^{(-)}(P) = \begin{pmatrix} \cos\Big(\vartheta^{(-)}(P)\Big) & \sin\Big(\vartheta^{(-)}(P)\Big) \\ -\sin\Big(\vartheta^{(-)}(P)\Big)\ \mathrm{e}^{-i\delta} & \cos\Big(\vartheta^{(-)}(P)\Big)\ \mathrm{e}^{-i\delta} \end{pmatrix} \tag{44}
$$

$$
\beta^{(+)}(P) = \begin{pmatrix} \cos\Big(\vartheta^{(+)}(P)\Big) & \sin\Big(\vartheta^{(+)}(P)\Big) \\ -\sin\Big(\vartheta^{(+)}(P)\Big)\ \mathrm{e}^{i\delta} & \cos\Big(\vartheta^{(+)}(P)\Big)\ \mathrm{e}^{i\delta} \end{pmatrix} \tag{45}
$$

The values of the mixing angles in matter $\vartheta^{(\pm)}(P)$, given by

$$
\tan\Big(2\vartheta^{(\pm)}(P)\Big) = \frac{\mp 2PV_C\sin(2\vartheta)}{\Delta \pm 2PV_C\cos(2\vartheta)} \tag{46}
$$

are the same as those obtained by diagonalizing the MSW equation in the mass basis.

4 Neutrino Oscillations

Let us consider first the neutrino oscillations in vacuum. In the weak basis the neutrino fields are coupled by the off-diagonal mass term, therefore it is not possible to solve the field equations and quantize the weak fields independently. This means that one cannot build a

Fock space of states for the flavour neutrinos. Nevertheless, for relativistic neutrinos, it is convenient to *define* the weak neutrino states $|\nu_\alpha\rangle$ as superpositions of mass eigenstates $|\nu_a\rangle$ with the Cabibbo-Kobayashi-Maskawa type mixing matrix $\mathcal{K}$

$$|\nu_\alpha\rangle \equiv \sum_a \mathcal{K}^*_{\alpha a} |\nu_a\rangle \tag{47}$$

where the mass eigenstates $|\nu_a\rangle$ are the states created from the vacuum by the independent operator fields in the mass basis. The weak states defined in Eq.(47) do not describe exactly the physical neutrinos produced or detected in weak interaction processes [17]. In order to elucidate this point, let us consider, for example, the weak charged current process $\nu_e + X_i \to e + X_f$ in which an electron neutrino is detected through the production of an electron. Through energy-momentum conservation, the amplitudes $\langle e|\bar{e}\gamma^\varrho(1+\gamma_5)\nu_a|\nu_a\rangle h_\varrho$ depend on the mass eigenvalues m_a (h_ϱ is the matrix element of the X part of the process). Therefore the amplitude

$$\langle e|\bar{e}\gamma^\varrho(1+\gamma_5)\nu_e|\nu_\alpha\rangle h_\varrho = \sum_a \mathcal{K}_{ea}\mathcal{K}^*_{\alpha a} \langle e|\bar{e}\gamma^\varrho(1+\gamma_5)\nu_a|\nu_a\rangle h_\varrho \neq 0 \tag{48}$$

even for $\alpha \neq e$. This is due to the fact that, even if the mixing matrix $\mathcal{K}$ satisfies $\mathcal{K}\mathcal{K}^\dagger = 1$, i.e. $\sum_a \mathcal{K}_{\alpha a}\mathcal{K}^*_{\beta a} = \delta_{\alpha\beta}$, the quantities $\langle e|\bar{e}\gamma^\varrho(1+\gamma_5)\nu_a|\nu_a\rangle h_\varrho$, which depend on the index a through the different masses m_a, spoil the diagonality of Eq.(48). In the limit in which the mass eigenvalues are negligible in comparison with the neutrino momentum, i.e. for relativistic neutrinos, the matrix elements can be taken out of the sum in Eq.(48) and the amplitude vanishes for $\alpha \neq e$. Therefore, the weak states defined in Eq.(47) describe only relativistic neutrinos produced or detected in weak interaction processes and will be called "relativistic weak states" in the following.

The non-diagonality shown in Eq.(48) implies that the usual formula for the neutrino oscillation amplitude

$$A_{\mu\to e}(t) = \sum_a \ \langle \nu_e|\ \nu_a\rangle\ \mathrm{e}^{-iE_a t}\ \langle \nu_a|\ \nu_\mu\rangle = \sum_a\ \mathcal{K}_{ea}\ \mathrm{e}^{-iE_a t}\ \mathcal{K}^*_{\mu a} \tag{49}$$

is not exact. In fact, since only the charged leptons can be detected, the correct formula is

$$A_{\mu\to e}(t) \sim \sum_a\ h^{(d)}_\varrho\ \langle e|\ \bar{e}\gamma^\varrho(1+\gamma_5)\nu_e\,|\nu_a\rangle\ \mathrm{e}^{-iE_a t}\ \left\langle \nu_a \right|\bar{\nu}_\mu\gamma^\lambda(1+\gamma_5)\mu\left|\mu\right\rangle\ h^{(p)}_\lambda \tag{50}$$

where the superscripts (p) and (d) for h_ϱ denote the production and detection processes, respectively. Inserting two complete set of relativistic weak states $\sum_\alpha |\nu_\alpha\rangle\langle\nu_\alpha| = 1$, one obtains

$$A_{\mu\to e}(t) \sim \sum_a\sum_{\alpha,\beta}\ h^{(d)}_\varrho\ \langle e|\ \bar{e}\gamma^\varrho(1+\gamma_5)\nu_e\,|\nu_\alpha\rangle\ \mathcal{K}_{\alpha a}\ \mathrm{e}^{-iE_a t}\ \mathcal{K}^*_{\beta a}\ \left\langle \nu_\beta \right|\bar{\nu}_\mu\gamma^\lambda(1+\gamma_5)\mu\left|\mu\right\rangle\ h^{(p)}_\lambda \tag{51}$$

As in Eq.(48), the matrix elements in Eq.(51) are not diagonal in the neutrino flavors and the total oscillation amplitude $A_{\mu\to e}(t)$ is different from the usual one given in Eq.(49). On the other hand, if the mass eigenvalues m_a are negligibly small in comparison with the neutrino momentum, i.e. for relativistic neutrinos, to lowest order Eq.(51) can be approximated to the usual oscillation amplitude given in Eq.(49). Therefore, the usual oscillation amplitude (49) describes correctly the neutrino oscillation phenomenology only for relativistic neutrinos.

At first sight the problem to define adequately the weak states in matter appears to be much more complicated than in vacuum, due to the fact that the neutrino fields do not mix in a simple way, but have a quite complicated structure, as shown in Eq.(32). However, one can note that the $(1+\gamma_5)$ factor in the weak charged current picks up only the coefficients $\alpha_{aj}^{(\pm)}(P)$ and in the relativistic approximation the positive helicity coefficients $\alpha_{aj}^{(+)}(P)$ are suppressed, whereas the negative helicity coefficients form a unitary matrix $\alpha^{(-)}(P)$. Therefore, following the same reasoning as in the vacuum case, the "relativistic weak states in matter" for negative helicity neutrinos (the positive helicity component of relativistic neutrinos is negligible, its relative contribution being of the order m_a/P) can be *defined* as

$$|\nu_\alpha^{(-)}(\vec{p})\rangle \equiv \sum_{a,j} \mathcal{K}_{\alpha a}^* \; \alpha_{aj}^{(-)*}(P) \; |\nu_j^{(-)}(\vec{p})\rangle \tag{52}$$

where $|\nu_j^{(-)}(\vec{p})\rangle$ are the energy eigenstates in matter created by the creation operators $a_j^\dagger(\vec{p},-)$. To lowest order in the relativistic approximation, the time evolution of the relativistic weak states in matter is given by

$$\left(i\frac{d}{dt} - P\right) |\nu_\alpha^{(-)}(\vec{p};t)\rangle = \frac{1}{2P} \sum_{j,\beta} \mathcal{S}_{\alpha j}^{(-)}(P) \; m_j^{(-)2} \; \mathcal{S}_{j\beta}^{(-)\dagger}(P) |\nu_\beta^{(-)}(\vec{p};t)\rangle \tag{53}$$

where $\mathcal{S}^{(-)}(P) \equiv \mathcal{K}^* \alpha^{(-)*}(P)$ is the effective mixing matrix between the relativistic weak states in matter and the energy eigenstates in matter, as can be seen in Eq.(52).

In the two generation case, the effective mixing matrix $\mathcal{S}^{(-)}(P)$ can be written as

$$\mathcal{S}^{(-)}(P) = \begin{pmatrix} \cos\left(\varphi^{(-)}(P)\right) & \sin\left(\varphi^{(-)}(P)\right) \\ -\sin\left(\varphi^{(-)}(P)\right) & \cos\left(\varphi^{(-)}(P)\right) \end{pmatrix} \tag{54}$$

where $\varphi^{(-)}(P) \equiv \vartheta + \vartheta^{(-)}(P)$ is the effective mixing angle in matter. The mixing matrix of the states do not depend on the CP violating phase δ and the value of the effective mixing angle $\varphi^{(-)}(P)$ is the same as the one obtained by diagonalizing the MSW time evolution equation, given in Eq.(3). The time evolution equation (53) of the weak states becomes

$$\left(i\frac{d}{dt} - P\right) \begin{pmatrix} |\nu_e^{(-)}(\vec{p};t)\rangle \\ |\nu_\mu^{(-)}(\vec{p};t)\rangle \end{pmatrix} = \frac{1}{4P} \left\{ \left(m_1^{(-)2} + m_2^{(-)2}\right) + \begin{pmatrix} -\Delta^{(-)}\cos\left(2\varphi^{(-)}(P)\right) & \Delta^{(-)}\sin\left(2\varphi^{(-)}(P)\right) \\ \Delta^{(-)}\sin\left(2\varphi^{(-)}(P)\right) & \Delta^{(-)}\cos\left(2\varphi^{(-)}(P)\right) \end{pmatrix} \right\} \begin{pmatrix} |\nu_e^{(-)}(\vec{p};t)\rangle \\ |\nu_\mu^{(-)}(\vec{p};t)\rangle \end{pmatrix} \tag{55}$$

where $\Delta^{(-)} = m_2^{(-)2} - m_1^{(-)2}$ is given by Eq.(3). Since from Eqs.(3) and (34)

$$\begin{aligned} \frac{1}{2}\left(m_1^{(-)2} + m_2^{(-)2}\right) &= 2PV_N + \frac{1}{2}\left[(m_1^2 + m_2^2) + 2PV_C\right] \\ \Delta^{(-)} \sin\left(2\varphi^{(-)}(P)\right) &= \Delta \sin(2\vartheta) \\ \Delta^{(-)} \cos\left(2\varphi^{(-)}(P)\right) &= \Delta \cos(2\vartheta) - 2PV_C \end{aligned} \tag{56}$$

then Eq.(55) leads to the MSW time evolution equation (2). The MSW equation is correct only in the lowest order of the relativistic approximation and in ordinary matter ($m_a \ll P$ and $V \ll P$). In the same approximation the neutrinos have negative helicity; this is why it is possible to derive the MSW equation by eliminating the spin structure in the Dirac equation, i.e. through a Klein-Gordon type equation [8].

We wish to comment on the fact that in our calculations the optical potentials which describe the matter effect are assumed to be constant, whereas the MSW equation is more interesting when the potentials change along the neutrino path since it allows a resonant flavour conversion. The constant optical potential approach is locally correct as long as the variation of the potential within a neutrino wavelength is negligible in comparison with the values of the parameters having dimension of energy (energy, momentum, masses and the potential itself). The relativistic weak states in matter, defined locally by Eq.(52), do not depend on how the neutrino has been produced or detected and represent the physical flavour neutrinos in the relativistic approximation. The connection between regions with different potentials can be made through the relativistic weak states, hence the MSW equation (2) holds even when the potentials are not constant.

5 Summary

In this paper we have presented the helicity formalism for the Majorana neutrino fields in matter in order to derive the effective spinorial wave functions for general (n, m) models. As an application, we have derived in a rigorous way the MSW equation that describes the neutrino oscillations in matter. In the usual MSW approach the energy eigenstates in matter are defined through the diagonalization of the weak states in matter. In a quantum field theoretical approach only the energy eigenstates in matter are well (and uniquely) defined through the solution of the field equations that allows one to build the Fock space of the energy eigenstates. Therefore, the rigorous approach to the neutrino oscillations in matter in quantum field theory is to define the weak states as appropriate superpositions of energy eigenstates (this can be done for relativistic neutrinos, as shown in Section 4) and derive the time evolution equation of the weak states from the well defined time evolution equation of the energy eigenstates. As shown in Section 4, the quantum field theoretical approach leads to the MSW equation only for relativistic neutrinos. For non-relativistic neutrinos the quantum field theoretical approach is necessary, since the flavour neutrinos are defined through their weak interactions with the charged leptons. However, for non-relativistic neutrinos the definition of the weak states is non-trivial, even in vacuum, since the production and detection amplitudes in weak interaction processes depend significantly on the mass eigenvalues. This issue and related topics will be discussed in detail elsewhere [17]. In paper II we will discuss the application of the helicity formalism to the majoron decays of neutrinos propagating in matter.

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