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CP Violation in the Lepton Sector for Majorana Neutrinos

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Abstract

The CP violation in the lepton sector for two generations of Majorana neutrinos is discussed with emphasis on the possibility of CP violation effects in the neutrino oscillations in a medium. It is shown that there can be some CP violation effect in the neutrino interactions in matter, but the effect does not appear in the oscillation probability calculated in the relativistic approximation.

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1 Introduction

A large class of grand unification models [1] suggest the existence of massive neutrinos, implying the possibility of neutrino mixings, oscillations and decays. This has given a new theoretical impetus to the investigation of the properties of neutrinos. Once neutrinos acquire mass, there can be some CP violation effect in the lepton sector due to the complex phase factors in the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix [2] for three or more generations of Dirac neutrinos. If the neutrinos are Majorana particles, there can be a CP violating phase in the weak mixing matrix even with only two generations of leptons [3, 4, 5].

In this paper we consider two generations of massive Majorana neutrinos and study some physical consequences of the CP violation phase that appears in the 2×2 unitary lepton mixing matrix.

First we review the argument that shows the presence of CP violation phase in the two generation Majorana case and summarize briefly the presence or absence of the CP violation phase in the neutrinoless double beta decay and the two generation neutrino oscillation in vacuum. We then study whether or not the CP phase will appear in the two generation neutrino oscillation in a medium. To this end we discuss the effective potential which neutrinos experience in matter and the distinction between the weak mixing of the fields and the states in matter, based on a helicity formalism for Majorana neutrino fields [6], and show that there can be some CP violation effect in the neutrino interactions in matter. However the effect does not appear in the oscillation probability calculated in the relativistic approximation.

The organization of this paper is as follows. In section 2 we discuss the mixing of Majorana neutrinos and show that the weak interaction Lagrangian is CP invariant when the CP violating phase has the values 0 and $\pi/2$ by deriving the CP conservation condition for the mixing matrix [7, 8]. In section 3 the CP violation effect in the neutrinoless double beta decay is reviewed, and in section 4 the CP violation effect in the neutrino oscillations in vacuum is investigated. In section 5, we derive the optical potentials in the effective weak interaction Lagrangian that describes the medium effect, which shows that the interactions with electrons in matter can in principle break CP invariance. In section 6 the weak mixing matrices of the fields and of the states are derived in the helicity formalism for the Majorana neutrinos in matter developed by Giunti, Kim and Lam [6]. In section 7 we discuss the possibility of CP violation effects in the Majorana neutrino oscillations in matter.

2 Mixing of Majorana Neutrinos

The Majorana mass term in the neutrino Lagrangian is given by

$$\mathcal{L}_M = -\frac{1}{2}\overline{(\nu_L)^c} M \nu_L - \frac{1}{2}\overline{\nu_L} M^\dagger (\nu_L)^c, \qquad \nu_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \end{pmatrix}, \tag{1}$$

where M is a 2×2 complex mass matrix, $\nu_{\alpha L}$ $(\alpha = e, \mu)$ are the left-handed chiral neutrino fields in the weak basis and $(\nu_{\alpha L})^c$ is the charge conjugate of $\nu_{\alpha L}$, given by

$$(\nu_{\alpha L})^c = \mathrm{C}\nu_{\alpha L}\mathrm{C}^{-1} = \mathcal{C}\overline{\nu_{\alpha L}}^T. \tag{2}$$

The charge conjugation matrix $\mathcal{C}$ is defined by

$$\mathcal{C}^{-1}\gamma^\mu\mathcal{C} = -(\gamma^\mu)^T, \qquad \mathcal{C} = -\mathcal{C}^T, \qquad \mathcal{C}^\dagger = \mathcal{C}^{-1}. \tag{3}$$

The complex mass matrix M is symmetric, $M^T = M$, since

$$\overline{(\nu_L)^c} M \nu_L = -(\nu_L)^T \mathcal{C}^{-1} M \nu_L = (\nu_L)^T M^T (\mathcal{C}^{-1})^T \nu_L = \overline{(\nu_L)^c} M^T \nu_L. \tag{4}$$

The mass matrix M can be diagonalized by a unitary transformation

$$U^T M U = M_D, \tag{5}$$

where U is a unitary matrix $(U^\dagger U = UU^\dagger = 1)$ and M_D is a diagonal matrix with real and positive definite eigenvalues $((M_D)_{kj} = m_k\delta_{kj},\ m_k \geq 0)$ [3, 5]. The mass eigenstate fields χ_L defined by

$$\nu_L = U\chi_L, \qquad \chi_L = U^\dagger \nu_L; \qquad \chi_L = \begin{pmatrix} \chi_{1L} \\ \chi_{2L} \end{pmatrix}, \tag{6}$$

diagonalize the Majorana mass Lagrangian:

$$\mathcal{L}_M = -\frac{1}{2}\overline{(\chi_L)^c} M_D \chi_L - \frac{1}{2}\overline{\chi_L} M_D (\chi_L)^c \tag{7}$$

By introducing the field χ:

$$\chi = \chi_L + (\chi_L)^c = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \tag{8}$$

the mass Lagrangian can be written as

$$\mathcal{L}_M = -\frac{1}{2}\bar{\chi}M_D\chi = -\frac{1}{2}\sum_k \bar{\chi}_k m_k \chi_k. \tag{9}$$

The fields χ_k $(k = 1, 2)$ are Majorana fields since they satisfy the self-conjugate condition

$$\chi^c = \chi. \tag{10}$$

The charged lepton current in the weak interaction Lagrangian is given by

$$j^\mu(x) = 2\,\overline{\ell_L}(x)\gamma^\mu\nu_L(x) = 2\,\overline{\ell_L}(x)\gamma^\mu U\chi_L(x) = 2\sum_{\alpha,k}\overline{\ell_{\alpha L}}(x)\gamma^\mu U_{\alpha k}\chi_{kL}(x) \tag{11}$$

where the two component column matrix $\ell_L(x)$ contains the left-handed projections of the charged lepton fields:

$$\ell_L(x) = \begin{pmatrix} e_L(x) \\ \mu_L(x) \end{pmatrix}. \tag{12}$$

For n generations the unitary matrix U contains n^2 real parameters of which $n(n-1)/2$ are rotation angles and $n^2 - n(n-1)/2 = n(n+1)/2$ are phases. The free Dirac Lagrangian of the charged lepton fields is invariant under a phase transformation

$$\ell_\alpha(x) \to e^{i\theta_\alpha}\ell_\alpha(x). \tag{13}$$

Hence the phases in the mixing matrix that can be eliminated by a transformation

$$U \to e^{-i\Theta}U; \qquad\qquad \Theta_{\alpha\beta} = \theta_\alpha\delta_{\alpha\beta}, \tag{14}$$

are not measurable quantities because they can be absorbed by a redefinition of the charged lepton fields. Instead the left-handed neutrino fields χ_L cannot absorb any phase factor because the Majorana mass Lagrangian (7) is not invariant under a phase transformation of the fields. Since there are n charged lepton fields, the number of physical phases in the unitary mixing matrix U is $n(n-1)/2 - n = n(n-1)/2$ [3, 4, 5]. This can be compared with the case in which the neutrinos are Dirac particles and the unitary mixing matrix contains $(n-1)(n-2)/2$ physical phases since a redefinition of the Dirac neutrino fields can absorb $n-1$ phase factors.

For two generations ($n = 2$), the mixing matrix U contains one physical phase; this is characteristic of the Majorana neutrinos. A 2×2 unitary matrix U can be written as

$$U = e^{i\delta} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\gamma} \end{pmatrix} \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix} . \tag{15}$$

The two phase factors on the left side can be eliminated by the transformation given in Eq.(14) with $\theta_e = \delta$ and $\theta_\mu = \delta + \gamma$. Then the mixing matrix U can be written in terms of two physical parameters, the mixing angle ϑ and a phase β, as

$$U = \begin{pmatrix} \cos\vartheta & \sin\vartheta \; e^{i\beta} \\ -\sin\vartheta & \cos\vartheta \; e^{i\beta} \end{pmatrix} . \tag{16}$$

The phase β is called "CP violating phase" because when $\beta \neq 0$ or $\beta \neq \pi/2$ the weak interaction Lagrangian is not CP invariant [9]. In order to proof this statement, we define the CP transformation of the neutrino fields χ_{kL} as

$$(\mathrm{CP})\chi_{kL}(x)(\mathrm{CP})^{-1} = \eta_k \gamma^0 \mathcal{C} \overline{\chi_{kL}}^T(x^P), \tag{17}$$

where $x^P = (x_0, -\vec{x})$ and η_k are phase factors called CP phases. In order to keep invariant the mass Lagrangian (7) under CP transformations, the CP phase factors η_k must be $\pm i$. The charged current weak interaction Lagrangian

$$\mathcal{L}_W^{CC}(x) = -\frac{g}{2\sqrt{2}} \Big[W_\mu^-(x) j^\mu(x) + W_\mu^+(x) j^{\mu\dagger}(x) \Big] \tag{18}$$

is invariant under CP transformation if the charged lepton current transforms under CP as

$$(\mathrm{CP}) j^\mu(x) (\mathrm{CP})^{-1} = \eta^{(W)} j_\mu^\dagger(x^P), \tag{19}$$

where $\eta^{(W)}$ is the CP phase of the gauge bosons $W^\pm$. Using Eq.(17), the charged lepton current transforms under CP as

$$(\mathrm{CP}) j^\mu(x) (\mathrm{CP})^{-1} = -2 \sum_{\alpha,k} \overline{\chi_{kL}}(x^P) \eta_\alpha^{(\ell)*} U_{\alpha k} \eta_k \gamma_\mu \ell_{\alpha L}(x^P). \tag{20}$$

where $\eta_\alpha^{(\ell)}$ are the CP phases of the charged lepton fields. Hence Eq.(19) is satisfied and CP is conserved [7, 8] if

$$\eta_\alpha^{(\ell)*} U_{\alpha k} \eta_k = \eta^{(W)} U_{\alpha k}^* . \tag{21}$$

The CP phases of the charged leptons are not measurable quantities since they can be eliminated by a phase transformation of the fields. Therefore the phases $\eta_\alpha^{(\ell)}$ can be chosen arbitrarily; in order to conserve CP in the one generation case one must choose $\eta^{(W)}\eta_\alpha^{(\ell)} = \pm i$. Let us choose $\eta^{(W)}\eta_\alpha^{(\ell)} = i$; then the CP conservation condition for the mixing matrix Eq.(21) can be written as

$$U_{\alpha k}^* = -iU_{\alpha k}\eta_k. \tag{22}$$

When $\beta = 0$ in the mixing matrix (16) the CP conservation condition (22) is satisfied for $\eta_1 = \eta_2 = +i$, namely the two mass eigenstate Majorana neutrino field have the same CP phase. The CP conservation condition (22) is also satisfied when $\beta = \pi/2$ by taking $\eta_1 = +i$ and $\eta_2 = -i$, namely the two mass eigenstate Majorana neutrino field have opposite CP phase. For other values of β the CP conservation condition (22) cannot be satisfied by any choice of the CP phases η_1 and η_2, therefore CP is not conserved by the charged current weak interactions.

3 Neutrinoless Double Beta Decay

In this section we discuss the CP violation effects in the neutrinoless double beta decay. The neutrinoless double beta decay is a second order process in the perturbative expansion of the charged current weak interactions in which a nucleus (A, Z) decays into a nucleus $(A, Z+2)$ with the emission of two electrons:

$$(A, Z) \to (A, Z+2) + e^- + e^-. \tag{23}$$

The neutrinoless double beta decay can occur if the neutrinos are massive Majorana particles. Since the nuclear energies are much smaller than the weak gauge boson mass, the neutrinoless double beta decay can be described by the effective low energy charged current weak Lagrangian

$$\begin{aligned}\mathcal{L}_{eff}^{CC}(x) &= -\frac{2G_F}{\sqrt{2}}\, \overline{e_L}(x)\gamma^\mu \nu_{eL}(x) J_\mu(x) + H.C. \\ &= -\frac{2G_F}{\sqrt{2}} \sum_k \overline{e_L}(x)\gamma^\mu U_{ek}\chi_{kL}(x) J_\mu(x) + H.C.\end{aligned} \tag{24}$$

where $J_\mu(x)$ is the hadronic current. The lepton part in the amplitude of the $(\beta\beta)_{0\nu}$ decay is proportional to

$$\sum_k (U_{ek})^2 \overline{u_e} \gamma^\mu \left(\frac{1+\gamma_5}{2}\right) \langle 0|T\left(\chi_{kL}(0)\ \chi_{kL}^T(0)\right)|0\rangle \left(\frac{1+\gamma_5}{2}\right) \gamma^{\nu T} \overline{u_e}^T . \tag{25}$$

Since the neutrino propagator is given by

$$\begin{aligned}\langle 0|T\left(\chi_{kL}(0)\ \chi_{kL}^T(0)\right)|0\rangle &= -\left(\frac{1+\gamma_5}{2}\right) \langle 0|T(\chi_k(0)\ \overline{\chi_k}(0))|0\rangle \left(\frac{1+\gamma_5}{2}\right) \mathcal{C} \\ &= -i \int \frac{d^4q}{(2\pi)^4} \left(\frac{1+\gamma_5}{2}\right) \frac{\gamma^\mu q_\mu + m_k}{q^2 - m_k^2} \left(\frac{1+\gamma_5}{2}\right) \mathcal{C} \\ &= -i \int \frac{d^4q}{(2\pi)^4} \frac{m_k}{q^2 - m_k^2} \left(\frac{1+\gamma_5}{2}\right) \mathcal{C},\end{aligned} \tag{26}$$

the lepton part in amplitude is proportional to

$$\sum_k (U_{ek})^2 \frac{m_k}{q^2 - m_k^2} \overline{u_e}(p_2) \gamma^\mu \left(\frac{1+\gamma_5}{2}\right) \gamma^\nu \mathcal{C} \overline{u_e}^T(p_1), \tag{27}$$

where q is the four-momentum of the virtual Majorana neutrino. Since the order of magnitude of the nuclear energies is much bigger than the neutrino masses, the m_k^2 in the denominator can be neglected and the $(\beta\beta)_{0\nu}$ decay rate is proportional to the square of the effective neutrino mass $\langle m_\nu \rangle$ given by

$$\begin{aligned}\langle m_\nu \rangle = \left|\sum_k (U_{ek})^2 m_k\right| &= \left|m_1 \cos^2\vartheta + m_2 \sin^2\vartheta e^{2i\beta}\right| \\ &= \left[(m_1 \cos^2\vartheta + m_2 \sin^2\vartheta)^2 - m_1 m_2 \sin^2(2\vartheta) \sin^2\beta\right]^{1/2} .\end{aligned} \tag{28}$$

The presence of the CP violating phase β can produce dramatic effects in the magnitude of the $(\beta\beta)_{0\nu}$ decay rate. For a fixed value of the neutrino masses m_1 and m_2 and the mixing angle ϑ, $\langle m_\nu \rangle$ is maximal for $\beta = 0$:

$$\langle m_\nu \rangle = m_1 \cos^2\vartheta + m_2 \sin^2\vartheta, \tag{29}$$

and minimal for $\beta = \pi/2$:

$$\langle m_\nu \rangle = m_1 \cos^2\vartheta - m_2 \sin^2\vartheta. \tag{30}$$

Note that for $\beta = \pi/2$ and $m_1/m_2 = \tan^2\vartheta$ the $(\beta\beta)_{0\nu}$ decay rate is zero. As explained in the previous section, in both of these cases CP is conserved and the cancellations in the $(\beta\beta)_{0\nu}$

decay rate is due to the fact that the two mass eigenstate Majorana neutrinos have opposite CP phase [9, 10, 8]. When β deviates from the values 0 and $\pi/2$, CP is not conserved and there is a partial cancellation in the $(\beta\beta)_{0\nu}$ decay rate between the contributions due to the propagators of the two mass eigenstate Majorana neutrinos.

4 Neutrino Oscillations in Vacuum

A relativistic neutrino ν_α $(\alpha = e, \mu, \tau)$ produced by a weak interaction process can be described by a superposition of mass eigenstates [6]

$$|\nu_\alpha(\vec{p})\rangle = \sum_k U^*_{\alpha k} |\chi_k(\vec{p})\rangle, \tag{31}$$

where the mixing matrix U for two generations is given by Eq.(16) and $|\chi_k(\vec{p})\rangle$ is the state of a mass eigenstate Majorana neutrino χ_k with momentum $\vec{p}$ and mass m_k. In the discussion of the neutrino oscillations we assume that the weak states $|\nu_\alpha(\vec{p})\rangle$ have a definite three-momentum $\vec{p}$, but their energy is not defined since each mass eigenstate has a different energy which, in the relativistic approximation $(m \ll |\vec{p}|)$, is given by

$$E_k = \sqrt{p^2 + m_k^2} \simeq p + \frac{m_k^2}{2p}; \qquad\qquad p \equiv |\vec{p}|. \tag{32}$$

The weak states given in Eq.(31) at the time $t = 0$ evolve in time as

$$|\nu_\alpha(\vec{p};t)\rangle = \sum_k e^{-iE_k t} U^*_{\alpha k} |\chi_k(\vec{p})\rangle. \tag{33}$$

In the relativistic approximation, the probability for the transition $\nu_\alpha \to \nu_\beta$ after a time t is

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta; t) = |\langle \nu_\beta(\vec{p}) | \nu_\alpha(\vec{p};t)\rangle|^2 &= \left| \sum_k U_{\beta k} e^{-iE_k t} U^*_{\alpha k} \right|^2 \\ &= \sum_k |U_{\beta k}|^2 |U_{\alpha k}|^2 + \sum_{k \neq j} U^*_{\beta k} U_{\alpha k} U_{\beta j} U^*_{\alpha j} \exp\left\{ i \frac{m_k^2 - m_j^2}{2p} t \right\}. \end{aligned} \tag{34}$$

The probability $P(\nu_\alpha \to \nu_\beta; t)$ is invariant under the phase transformation of the mixing matrix

$$U_{\alpha k} \to U_{\alpha k} e^{i\rho_k}, \tag{35}$$

where ρ_k are arbitrary real parameters. The physical $n-1$ phases contained in the mixing matrix U that are characteristic of n generations of Majorana neutrinos, in comparison with the case of n generations of Dirac neutrinos, can be cancelled by the phase transformation given in Eq.(35). Therefore the neutrino flavour oscillations depend on only $(n-1)(n-2)/2$ phases, independently of the Majorana or Dirac character of the neutrinos, namely it is impossible to distinguish Majorana and Dirac neutrinos in flavour oscillation experiments [4, 5]. For two generations the transition probability is given by

$$P(\nu_\alpha \to \nu_\beta; R) = \sin^2(2\vartheta)\sin^2\left(\frac{\Delta}{4p}R\right); \qquad \alpha \neq \beta \tag{36}$$

where R is the length of flight that for relativistic neutrino is equivalent to the time of flight ($R = t$) and $\Delta = m_2^2 - m_1^2$. The transition probability given in Eq.(36) is oscillating as a function of the distance with oscillation length $4\pi E/\Delta$ and does not depend on the CP violating phase β characteristic of Majorana neutrinos. Therefore there is no CP violation effect in the two generation neutrino flavor oscillations in vacuum.

The neutrino flavor oscillation is due to the interference of the wave functions of the different mass eigenstates that propagate with different energies, namely it is a kinematical effect that occurs independently of the fact that neutrinos are Dirac or Majorana particles (the total lepton number is conserved). In order to see if the CP violating phases characteristic of Majorana neutrinos are physically observable quantities in some neutrino oscillation experiment, one must consider a lepton number non conserving process. For example [11], consider a neutrino produced at a time $t = 0$ by the interaction of a positively charged lepton with a neutron $\ell_\alpha^+ + n \to \nu_\alpha + p$; the neutrino ν_α is a superposition of mass eigenstate neutrinos χ_k that propagate with different energies E_k ; after a time t the neutrino $\nu_\alpha(t)$ is detected by the process $\nu_\alpha(t) + n \to \ell_\beta^- + p$ in which a negatively charged lepton is produced. The overall process violates the total lepton number conservation ($\Delta L = 2$) and is analog to the neutrinoless double β^+ decay, with the difference that a real neutrino, instead of a virtual one, propagate between the two interaction vertices. The amplitude for the overall process is given by the sum over the index k, that denotes the mass eigenstates, of the product of the following three amplitudes:

1. The amplitude $A_1^{(k)}$ for the interaction $\ell_\alpha^+ + n \to \chi_k + p$:

$$A_1^{(k)} = \frac{G_F}{\sqrt{2}} \langle \chi_k | \overline{\ell_\alpha}(0) \gamma^\lambda (1+\gamma_5) U_{\alpha k} \chi_k(0) | \ell_\alpha^+ \rangle \; h_{1\lambda}^{(k)}, \tag{37}$$

where $h_{1\lambda}^{(k)}$ is the hadronic matrix element. This interaction is allowed only if the neutrinos are Majorana particles, namely $\chi = \chi^c$, in which case

$$A_1^{(k)} = -\frac{G_F}{\sqrt{2}} \, \overline{v_{\ell_\alpha}} \gamma^\lambda (1+\gamma_5) U_{\alpha k} v_{\chi_k} h_{1\lambda}^{(k)}. \tag{38}$$

2. The amplitude for the mass eigenstate neutrino χ_k to propagate from the initial time $t=0$ to the final time t:

$$e^{-iE_k t} \simeq e^{-ipt} \exp\left\{ -i \frac{m_k^2}{2p} t \right\}. \tag{39}$$

3. The amplitude $A_2^{(k)}$ for the interaction $\chi_k + n \to \ell_\beta^- + p$:

$$\begin{aligned} A_2^{(k)} &= \frac{G_F}{\sqrt{2}} \langle \ell_\beta^- | \overline{\ell_\beta}(0) \gamma^\rho (1+\gamma_5) U_{\beta k} \chi_k(0) | \chi_k \rangle \; h_{2\rho}^{(k)} \\ &= \frac{G_F}{\sqrt{2}} \, \overline{u_{\ell_\beta}} \gamma^\rho (1+\gamma_5) U_{\beta k} u_{\chi_k} h_{2\rho}^{(k)}. \end{aligned} \tag{40}$$

Since in the relativistic approximation the k dependence of the charged lepton and the hadron parts of the amplitudes (that occurs through the energy-momentum conservation in the interaction vertices) is negligible, the total amplitude is proportional to

$$\sum_k U_{\beta k} U_{\alpha k} e^{-iE_k t} (1+\gamma_5) \frac{\not{p}_k + m_k}{E_k} (1+\gamma_5) \sim \sum_k \frac{m_k}{p} U_{\beta k} U_{\alpha k} \exp\left\{ -i \frac{m_k^2}{2p} t \right\}. \tag{41}$$

Hence the probability of the lepton number non-conserving process under consideration is proportional to

$$\begin{aligned} P_{\bar{\alpha}\to\beta}(t) &\sim \left| \sum_k \frac{m_k}{p} U_{\beta k} U_{\alpha k} \exp\left\{ -i \frac{m_k^2}{2p} t \right\} \right|^2 \\ &\sim \sum_k \left(\frac{m_k}{p} \right)^2 |U_{\beta k} U_{\alpha k}|^2 + \sum_{j \neq k} \left(\frac{m_k m_j}{p^2} \right) U_{\beta k}^* U_{\alpha k}^* U_{\beta j} U_{\alpha j} \exp\left\{ i \frac{m_k^2 - m_j^2}{2p} t \right\}. \end{aligned} \tag{42}$$

In the literature this lepton number non-conserving process has been sometimes called "neutrino-antineutrino oscillation" [12] because if the neutrinos are Dirac particles the first interaction creates an antineutrino; we believe that this statement is highly misleading for two reasons: 1) a lepton number non-conserving processes is possible only if the neutrinos are Majorana particles, in which case there is no distinction between particle and antiparticle; 2) during the propagation the mass eigenstates χ_k do not change except for the kinematical phase factors $\exp\{-iE_k t\}$ that produces the usual flavour neutrino oscillation. In the lepton number non-conserving process under consideration there is a mismatch between the helicity of the propagating neutrino and the chiral character of one of the two interaction vertices that carry a suppression factor of order m^2/p^2, as in the case of the neutrinoless double beta decay. Therefore this process should be called "lepton number non-conserving flavour neutrino oscillation". For two generations the probabilities $P_{\bar{e}\to e}(R)$ and $P_{\bar{e}\to\mu}$ are given by [11]

$$P_{\bar{e}\to e}(R) \sim \frac{m_1 m_2}{p^2}\left\{\left(\frac{m_1}{m_2}\cos^4\vartheta + \frac{m_2}{m_1}\sin^4\vartheta\right) + \frac{1}{2}\sin^2(2\vartheta)\cos\left(\frac{\Delta}{2p}R + 2\beta\right)\right\} \quad (43)$$

$$P_{\bar{e}\to\mu}(R) \sim \frac{m_1 m_2}{4p^2}\sin^2(2\vartheta)\left\{\left(\frac{m_1}{m_2} + \frac{m_2}{m_1}\right) - 2\cos\left(\frac{\Delta}{2p}R + 2\beta\right)\right\}. \quad (44)$$

Therefore, the CP violating phases characteristic of the Majorana neutrinos are observable in the lepton number non-conserving flavour neutrino oscillations, but the experimental observation of these processes is practically impossible because of the strong suppression factor m^2/p^2. As explained at the end of section 2, if CP is conserved then $\beta = 0$ if the two mass eigenstate Majorana neutrinos have the same CP phase and $\beta = \pi/2$ if they have opposite CP phase. Therefore, from Eq.s(43) and (44), the relative sign of the CP phases of the two mass eigenstate Majorana neutrinos can be in principle measured in the lepton number non-conserving flavour neutrino oscillations, as in the neutrinoless double beta decays.

5 Effective potentials in matter

In this section we derive the value of the optical potentials V_C and V_N in the effective weak interaction Lagrangian that describes the medium effect. The optical potentials are due to the coherent charged current (V_C) and neutral current (V_N) weak interactions with the particles of the medium. The coherence requirement allows one to write the weak interaction

Lagrangian only in terms of the neutrino field and the medium effect is described by an optical potential that depends on the matter density and composition. Therefore the neutrino field equation is decoupled from the many-body field equations of the medium and can be solved. Since the weak charged current $j_\mu^{CC}(x)$ is given by

$$j_\mu^{CC}(x) = 2 \ \overline{e_L}(x)\gamma_\mu \nu_{eL}(x), \tag{45}$$

the effective weak charged current interaction Lagrangian can be written as

$$\mathcal{L}_{eff}^{CC(e)}(x) = -\frac{4G_F}{\sqrt{2}} \int d^3p_e f(E_e,T) \langle e(p_e)|\overline{e_L}(x)\gamma_\mu \nu_{eL}(x)\overline{\nu_{eL}}(x)\gamma^\mu e_L(x)|e(p_e)\rangle \tag{46}$$

where $f(E_e, T)$ is the statistical energy distribution function of the electrons in a homogeneous and isotropic medium with temperature T (we work in the rest frame of the medium) and it is normalized to

$$\int d^3p_e f(E_e,T) = 1. \tag{47}$$

The electron part of the matrix element in Eq.(46) can be separated out by a Fierz transformation and is given by

$$\begin{aligned}
\langle e(p_e)|\overline{e_L}(x)\gamma^\mu e_L(x)|e(p_e)\rangle &= \overline{u}(p_e)\gamma^\mu \frac{1+\gamma_5}{2} u(p_e) \\
&= \frac{N_e}{2} Tr \left\{ \frac{\not{p}_e + m_e}{2E_e} \gamma^\mu \frac{1+\gamma_5}{2} \right\} \\
&= N_e \frac{p_e^\mu}{2E_e}
\end{aligned} \tag{48}$$

where N_e is the electron number density and we have averaged over the electron spin states. The Dirac spinors have been normalized to

$$u^\dagger(p,s)u(p,s') = v^\dagger(p,s)v(p,s') = \delta_{ss'}. \tag{49}$$

With Eq.(48), the integral factor in Eq.(46) reduces to

$$\int d^3p_e f(E_e,T) \frac{\gamma_\mu p_e^\mu}{E_e} = \int d^3p_e f(E_e,T) \left(\gamma_0 - \frac{\vec{\gamma}\cdot\vec{p_e}}{E_e} \right) = \gamma_0. \tag{50}$$

Hence, we obtain

$$\mathcal{L}_{eff}^{CC(e)}(x) = -V_C \ \overline{\nu_{eL}}(x)\gamma^0 \nu_{eL}(x) \qquad \text{where} \qquad V_C = \frac{G_F}{\sqrt{2}} 2N_e. \tag{51}$$

The effective charged current weak interaction Lagrangian does not depend on the temperature of the medium, as long as the medium is isotropic and to the extent that the Fermi four point interaction is valid.

In the chiral representation for the γ -matrices:

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \tag{52}$$

a Majorana electron neutrino field ν_e can be written as

$$\nu_e = \begin{pmatrix} \Phi_e \\ -i\sigma^2\Phi_e^* \end{pmatrix} \tag{53}$$

where $\Phi_e(x)$ is a two-component spinor field. Then the effective weak charged current interaction Lagrangian can be written as

$$\mathcal{L}_{eff}^{CC(e)}(x) = -V_C\ \Phi_e^\dagger\Phi_e. \tag{54}$$

In a neutral medium the effective weak neutral current interaction Lagrangian is due to the interaction of the electron neutrino with the neutrons of the medium, since the contributions of the weak neutral current interactions with the electrons and the protons cancel each other. Following an approach similar to that used to derive $\mathcal{L}_{eff}^{CC(e)}(x)$, we obtain

$$\mathcal{L}_{eff}^{NC(e)}(x) = -V_N\ \Phi_e^\dagger\Phi_e; \qquad\qquad V_N = -\frac{G_F}{\sqrt{2}}N_n \tag{55}$$

where N_n is the neutron number density.

Thus the total effective weak interaction Lagrangian for a Majorana electron neutrino is

$$\mathcal{L}_{int}^{(e)}(x) = -V_e\ \Phi_e^\dagger\Phi_e, \tag{56}$$

where the optical potential for the electron neutrino field is

$$V_e = V_C + V_N = \frac{G_F}{\sqrt{2}}(2N_e - N_n). \tag{57}$$

The optical potential for the muon neutrino field is given by

$$V_\mu = V_N = -\frac{G_F}{\sqrt{2}}N_n. \tag{58}$$

Hence the effective weak interaction Lagrangian, that describes the medium effects for two generations in the weak basis, is

$$\mathcal{L}_{int}(x) = -\Phi^\dagger(x) V \Phi(x); \qquad \Phi = \begin{pmatrix} \Phi_e(x) \\ \Phi_\mu(x) \end{pmatrix} \tag{59}$$

where the effective potential matrix is given by

$$V = \begin{pmatrix} V_C + V_N & 0 \\ 0 & V_N \end{pmatrix} \tag{60}$$

The two-component spinor field in the mass basis Φ_M is related with the two-component spinor field in the weak basis Φ by the unitary transformation

$$\Phi = U\Phi_M; \qquad \Phi_M = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \tag{61}$$

where U is the unitary mixing matrix given in Eq.(16). Thus the interaction Lagrangian $\mathcal{L}_{int}(x)$ in the mass basis is

$$\mathcal{L}_{int}(x) = -\Phi_M^\dagger(x) V_M \Phi_M(x) \tag{62}$$

where the effective potential matrix V_M is given by

$$V_M = U^\dagger V U = \begin{pmatrix} V_N + V_C\cos^2\vartheta & V_C \sin\vartheta\cos\vartheta e^{i\beta} \\ & \\ V_C\sin\vartheta\cos\vartheta e^{-i\beta} & V_N + V_C\sin^2\vartheta \end{pmatrix}. \tag{63}$$

The effective potential matrix in the mass basis V_M depends on the mixing angle ϑ and on the CP violating phase β. The fact that the phase factor β appears in the off-diagonal terms together with V_C shows that the interactions with the electrons in matter can in principle break the CP invariance of the Majorana neutrino Lagrangian.

6 Majorana Neutrino Fields in Matter

The effective Lagrangian for two generations of Majorana neutrino fields in the mass basis can be written as

$$\begin{aligned} \mathcal{L}_{eff}(x) &= \frac{1}{2}\Phi_M^\dagger(x) i\bar{\sigma}^\mu \partial_\mu \Phi_M(x) \qquad (64) \\ &- \frac{1}{2}\left[\Phi_M^T(x) i\sigma^2 M_D \Phi_M(x) - \Phi_M^\dagger(x) M_D i\sigma^2 \Phi_M^*(x)\right] - \frac{1}{2}\Phi_M^\dagger(x) V_M \Phi_M(x), \end{aligned}$$

where $\overline{\sigma}^\mu = (1, -\vec{\sigma})$. From the effective Lagrangian we obtain the field equation

$$(i\overline{\sigma}^\mu \partial_\mu - V_M)\Phi_M(x) + M_D i\sigma^2 \Phi^*_M(x) = 0. \tag{65}$$

Since the helicity is conserved by the weak interactions of the neutrino with the medium, we can solve the field equation by expanding the field $\Phi_M(x)$ as a superposition of plane wave spinors with definite helicity:

$$\Phi_M(x) = \int d^3\tilde{p}\; e^{i\vec{p}\cdot\vec{x}} \Big\{ \left[A^{(+)}_M(\vec{p},t) + B^{(+)}_M(\vec{p},t)\right] \omega(\vec{p},+) + \left[A^{(-)}_M(\vec{p},t) + B^{(-)}_M(\vec{p},t)\right] \omega(\vec{p},-) \Big\}, \tag{66}$$

where $d^3\tilde{p} = \frac{d^3p}{\sqrt{(2\pi)^3}}$ and $A^{(\pm)}_M(\vec{p},t)$ and $B^{(\pm)}_M(\vec{p},t)$ are the positive and negative frequency parts of the fields, the superscript indicates the helicity and $\omega(\vec{p},\mp)$ are the two-component ortho-normalized helicity eigenstate spinors:

$$\omega(\vec{p},+) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)\, e^{i\phi} \end{pmatrix} \qquad \omega(\vec{p},-) = \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right)\, e^{-i\phi} \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \tag{67}$$

where $\vec{p} = (p, \theta, \phi)$ in the polar coordinate system. They satisfy the following useful relations ($p \equiv |\vec{p}\,|$):

$$\frac{\vec{\sigma}\cdot\vec{p}}{p}\, \omega(\vec{p},\mp) = \pm\omega(\vec{p},\mp) \tag{68}$$

$$-i\sigma^2\omega^*(\vec{p},\pm) = \pm\omega(\vec{p},\mp) \tag{69}$$

$$\omega(-\vec{p},\pm) = \mp e^{\pm i\phi}\omega(\vec{p},\mp). \tag{70}$$

The field equation (65) yields the equation

$$(i\partial_0 - p - V_M)\left[A^{(-)}_M(\vec{p},t) + B^{(-)}_M(\vec{p},t)\right] - M_D e^{i\phi}\left[A^{(-)*}_M(-\vec{p},t) + B^{(-)*}_M(-\vec{p},t)\right] = 0 \tag{71}$$

for the negative helicity sector; since the analogous equation for the positive helicity sector can be obtained by replacing $p \to -p$, hereafter we will consider only the negative helicity sector. The above matrix equation can be written in the component form as

$$\begin{aligned}(i\partial_0 - p - V_{11})\left[A^{(-)}_1(\vec{p},t) + B^{(-)}_1(\vec{p},t)\right] - V_{12}e^{i\beta}\left[A^{(-)}_2(\vec{p},t) + B^{(-)}_2(\vec{p},t)\right]& \\ - m_1 e^{i\phi}\left[A^{(-)*}_1(-\vec{p},t) + B^{(-)*}_1(-\vec{p},t)\right] = 0&\end{aligned} \tag{72}$$

$$(i\partial_0 - p - V_{22})\left[A_2^{(-)}(\vec{p},t) + B_2^{(-)}(\vec{p},t)\right] - V_{12}e^{-i\beta}\left[A_1^{(-)}(\vec{p},t) + B_1^{(-)}(\vec{p},t)\right]$$
$$- m_2 e^{i\phi}\left[A_2^{(-)*}(-\vec{p},t) + B_2^{(-)*}(-\vec{p},t)\right] = 0 \tag{73}$$

where V_{ij} are the matrix elements of V_M. In order to quantize properly the fields $\Phi_k(x)$, we expand the positive and negative frequency parts $A_k^{(-)}(\vec{p},t)$ and $B_k^{(-)}(\vec{p},t)$ as a superposition of energy eigenstates as

$$A_k^{(-)}(\vec{p},t) = \sum_{n=1,2} \alpha_{kn}^{(-)}(p) a_n(\vec{p},-)\ e^{-iE_n^{(-)}t} \tag{74}$$

$$e^{i\phi}B_k^{(-)*}(-\vec{p},t) = \sum_{n=1,2} \beta_{kn}^{(-)}(p) a_n(\vec{p},-)\ e^{-iE_n^{(-)}t} \tag{75}$$

where $E_n^{(-)}$ are the energy eigenvalues of the negative helicity states (the sum over n goes from 1 to 2 because there are two degrees of freedom for each helicity eigenvalue). $a_1(\vec{p},-)$ and $a_2(\vec{p},-)$ are operators that obey the canonical anticommutation relations and can be interpreted as destruction operators for the negative helicity eigenstates [6]. The values of the coefficients $\alpha_{kn}^{(-)}(p)$ and $\beta_{kn}^{(-)}(p)$ are given by the eigenvalue equation:

$$\begin{pmatrix} (p+V_{11}) & m_1 & V_{12}e^{i\beta} & 0 \\ m_1 & -(p+V_{11}) & 0 & -V_{12}e^{-i\beta} \\ V_{12}e^{-i\beta} & 0 & (p+V_{22}) & m_2 \\ 0 & -V_{12}e^{i\beta} & m_2 & -(p+V_{22}) \end{pmatrix} \begin{pmatrix} \alpha_{1n}^{(-)}(p) \\ \beta_{1n}^{(-)}(p) \\ \alpha_{2n}^{(-)}(p) \\ \beta_{2n}^{(-)}(p) \end{pmatrix} = E_n^{(-)} \begin{pmatrix} \alpha_{1n}^{(-)}(p) \\ \beta_{1n}^{(-)}(p) \\ \alpha_{2n}^{(-)}(p) \\ \beta_{2n}^{(-)}(p) \end{pmatrix} \tag{76}$$

The secular equation of Eq.(76) is quadratic in $E_n^{(-)^2}$ and gives two positive energy eigenvalues:

$$E_n^{(-)2} = \frac{1}{2}\left\{\left[(p+V_N)^2 + (p+V_N+V_C)^2 + \Sigma\right] \pm \Delta^{(-)}\right\} \tag{77}$$

where $\Sigma \equiv m_1^2 + m_2^2$ and

$$\Delta^{(-)} = \left\{\left[\Delta\cos(2\vartheta) - 2V_C(p+V_N)\right]^2 + \left[\Delta\sin(2\vartheta)\right]^2 \right. \tag{78}$$
$$\left. + V_C^2\left[V_C^2 + 4(p+V_N)V_C - 2\Delta\cos(2\vartheta) + \Sigma\sin^2(2\vartheta) - 2m_1m_2\sin^2(2\vartheta)\cos 2\beta\right]\right\}^{1/2}.$$

The energy eigenvalues depend upon the CP violating phase β, but for a relativistic neutrino the dependence is suppressed by a factor of order m/p in comparison with the leading term.

From the second and fourth row of Eq.(76), we obtain

$$\begin{aligned}
\beta_{1n}^{(-)} &= \frac{m_1(E_n^{(-)}+p+V_{22})\alpha_{1n}^{(-)}(p) - m_2 V_{12} e^{-i\beta}\alpha_{2n}^{(-)}(p)}{(E_n^{(-)}+p_N)(E_n^{(-)}+p_C)} \\
\beta_{2n}^{(-)} &= \frac{m_2(E_n^{(-)}+p+V_{11})\alpha_{2n}^{(-)}(p) - m_1 V_{12} e^{i\beta}\alpha_{1n}^{(-)}(p)}{(E_n^{(-)}+p_N)(E_n^{(-)}+p_C)}
\end{aligned} \tag{79}$$

where we have introduced the notations

$$p_N = p + V_N, \qquad\qquad p_C = p + V_N + V_C. \tag{80}$$

These equations allow us to decouple the equations for the positive and negative frequency components of the fields: by elimination of $\beta_{1n}^{(-)}(p)$ and $\beta_{2n}^{(-)}(p)$, Eq.(76) reduces to

$$\begin{pmatrix} E_n^{(-)} - (p+V_{11}) - q_{11} & -V_{12}e^{i\beta} + q_{12}e^{-i\beta} \\ -V_{12}e^{-i\beta} + q_{12}e^{i\beta} & E_n^{(-)} - (p+V_{22}) - q_{22} \end{pmatrix} \begin{pmatrix} \alpha_{1n}^{(-)}(p) \\ \alpha_{2n}^{(-)}(p) \end{pmatrix} = 0 \tag{81}$$

where we have defined for simplicity

$$\begin{aligned}
q_{11} &= \frac{m_1^2(E_n^{(-)}+p+V_{22})}{(E_n^{(-)}+p_N)(E_n^{(-)}+p_C)} \\
q_{12} &= \frac{m_1 m_2 V_{12}}{(E_n^{(-)}+p_N)(E_n^{(-)}+p_C)} \\
q_{22} &= \frac{m_2^2(E_n^{(-)}+p+V_{11})}{(E_n^{(-)}+p_N)(E_n^{(-)}+p_C)}.
\end{aligned} \tag{82}$$

Equation (81) gives the ratio between the $\alpha_{kn}^{(-)}(p)$ $(k=1,2)$ and the overall factor in the solutions can be fixed by their normalization. The plane wave expansion of the two-component Majorana fields $\Phi_k(x)$ is given by

$$\begin{aligned}
\Phi_k(x) = \int d^3\tilde{p} \sum_{n=1,2} \Big\{ & \alpha_{kn}^{(-)}(p)\ \omega(\vec{p},-)\ a_n(\vec{p},-)\ e^{-iE_n^{(-)}t + i\vec{p}\cdot\vec{x}} \\
& + \alpha_{kn}^{(+)}(p)\ \omega(\vec{p},+)\ a_n(\vec{p},+)\ e^{-iE_n^{(+)}t + i\vec{p}\cdot\vec{x}} \\
& + \beta_{kn}^{(-)*}(p)\ \omega(\vec{p},+)\ a_n^\dagger(\vec{p},-)\ e^{iE_n^{(-)}t - i\vec{p}\cdot\vec{x}} \\
& - \beta_{kn}^{(+)*}(p)\ \omega(\vec{p},-)\ a_n^\dagger(\vec{p},+)\ e^{iE_n^{(+)}t - i\vec{p}\cdot\vec{x}} \Big\},
\end{aligned} \tag{83}$$

where the coefficients $\beta_{kn}^{(+)}(p)$ can be obtained by replacing $p \to -p$ in Eq.(76). The four-component Majorana neutrino field $\chi_k(x)$ is given by

$$\chi_k(x) = \int d^3\tilde{p} \sum_{n=1,2} \left\{ \begin{pmatrix} \alpha_{kn}^{(-)}(p)\ \omega(\vec{p},-) \\ \beta_{kn}^{(-)}(p)\ \omega(\vec{p},-) \end{pmatrix} a_n(\vec{p},-)\ e^{-iE_n^{(-)}t+i\vec{p}\cdot\vec{x}} \right.$$
$$+ \begin{pmatrix} \alpha_{kn}^{(+)}(p)\ \omega(\vec{p},+) \\ \beta_{kn}^{(+)}(p)\ \omega(\vec{p},+) \end{pmatrix} a_n(\vec{p},+)\ e^{-iE_n^{(+)}t+i\vec{p}\cdot\vec{x}}$$
$$+ \begin{pmatrix} \beta_{kn}^{(-)*}(p)\ \omega(\vec{p},+) \\ -\alpha_{kn}^{(-)*}(p)\ \omega(\vec{p},+) \end{pmatrix} a_n^\dagger(\vec{p},-)\ e^{iE_n^{(-)}t-i\vec{p}\cdot\vec{x}}$$
$$\left. + \begin{pmatrix} -\beta_{kn}^{(+)*}(p)\ \omega(\vec{p},-) \\ \alpha_{kn}^{(+)*}(p)\ \omega(\vec{p},-) \end{pmatrix} a_n^\dagger(\vec{p},+)\ e^{iE_n^{(+)}t-i\vec{p}\cdot\vec{x}} \right\}. \tag{84}$$

This equation shows that in matter the components of the neutrino fields do not mix in a simple way as one would have naively expected.

Since the weak states are defined as a mixing of energy eigenstates only in the relativistic approximation [6], it is useful to see what is the value of the mixing parameters $\alpha_{kn}^{(-)}(p)$ for $m^2 \sim pV \ll p^2$. To order $m^2/p^2 \sim V/p$ we can approximate

$$E_n^{(-)} = p + \frac{m_n^{(-)^2}}{2p} \tag{85}$$

where $m_n^{(-)2}$ are the effective squared masses in matter. From Eq.s(77) and (79) the energy eigenvalues $E_n^{(-)}$ are approximated by

$$E_n^{(-)} = p + V_N + \frac{1}{4p}\left[\Sigma + 2pV_C\right] \pm \frac{1}{p}\Delta^{(-)} \tag{86}$$

$$\Delta^{(-)} = \sqrt{\left[\Delta\cos(2\vartheta) - 2pV_C\right]^2 + \left[\Delta\sin(2\vartheta)\right]^2}. \tag{87}$$

The eigenvalue equation (81) can be approximated to

$$\begin{pmatrix} \frac{m_n^{(-)^2}-m_1^2}{2p} - V_{11} & -V_{12}e^{i\beta} \\ -V_{12}e^{-i\beta} & \frac{m_n^{(-)^2}-m_2^2}{2p} - V_{22} \end{pmatrix} \begin{pmatrix} \alpha_{1n}^{(-)}(p) \\ \alpha_{2n}^{(-)}(p) \end{pmatrix} = 0. \tag{88}$$

The eigenvalues are

$$m_n^{(-)^2} = 2pV_N + \frac{1}{2}\left[\Sigma + 2pV_C\right] \pm \frac{1}{2}\Delta^{(-)}. \tag{89}$$

The solutions can be parametrized in terms of a mixing angle $\vartheta^{(-)}(p)$ and the CP violating phase β as

$$\begin{pmatrix} \alpha_{11}^{(-)}(p) \\ \alpha_{21}^{(-)}(p) \end{pmatrix} = \begin{pmatrix} \cos\left(\vartheta^{(-)}(p)\right) \\ -\sin\left(\vartheta^{(-)}(p)\right) e^{-i\beta} \end{pmatrix}, \qquad \begin{pmatrix} \alpha_{12}^{(-)}(p) \\ \alpha_{22}^{(-)}(p) \end{pmatrix} = \begin{pmatrix} \sin\left(\vartheta^{(-)}(p)\right) \\ \cos\left(\vartheta^{(-)}(p)\right) e^{-i\beta} \end{pmatrix}. \tag{90}$$

The mixing angle $\vartheta^{(-)}(p)$ is given by

$$\tan\left(2\vartheta^{(-)}(p)\right) = \frac{2pV_C \sin(2\vartheta)}{\Delta - 2pV_C \cos(2\vartheta)} \tag{91}$$

Therefore the relativistic approximation the negative helicity components of the neutrino fields mix in a simple way through the unitary mixing matrix:

$$\alpha^{(-)}(p) = \begin{pmatrix} \cos\left(\vartheta^{(-)}(p)\right) & \sin\left(\vartheta^{(-)}(p)\right) \\ -\sin\left(\vartheta^{(-)}(p)\right) e^{-i\beta} & \cos\left(\vartheta^{(-)}(p)\right) e^{-i\beta} \end{pmatrix}. \tag{92}$$

7 Neutrino Oscillation in Matter

In vacuum a relativistic neutrino produced by a weak interaction process is described by a superposition of mass eigenstates as given in Eq.(31). In matter the weak states can be defined in terms of matter energy eigenstates as

$$|\nu_\alpha(\vec{p},-)\rangle = \sum_{n=1,2} S_{\alpha n}^{(-)}(p)|\chi_n(\vec{p},-)\rangle \qquad (\alpha = e,\mu) \tag{93}$$

where $|\chi_n(\vec{p},-)\rangle$ $(n = 1,2)$ are the energy eigenstates in matter created by the creation operators $a_n^\dagger(\vec{p},-)$. The unitary matrix $S^{(-)}(\vec{p})$ depends on the neutrino momentum and on the matter density.

Since the energy eigenstate evolve in time according to

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\chi_n(\vec{p},-;t)\rangle = H|\chi_n(\vec{p},-;t)\rangle = E_n^{(-)}|\chi_n(\vec{p},-;t)\rangle, \tag{94}$$

the time evolution equation of the weak states is

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\nu_\alpha(\vec{p},-;t)\rangle = \sum_{n,\beta} S^{(-)}_{\alpha n}(p)E^{(-)}_n S^{(-)*}_{\beta n}(p)|\nu_\beta(\vec{p},-;t)\rangle. \tag{95}$$

The relativistic approximation to order $m^2/P^2 \sim V/P$ yields

$$\left(i\frac{\mathrm{d}}{\mathrm{d}t} - p\right)|\nu_\alpha(\vec{p},-;t)\rangle = \frac{1}{2p}\sum_{n,\beta} S^{(-)}_{\alpha n}(p)m^{(-)2}_n S^{(-)*}_{\beta n}(\vec{p})|\nu_\beta(\vec{p},-;t)\rangle. \tag{96}$$

In this equation we need to know the values of the unitary mixing matrix only to zeroth order in $m^2/p^2 \sim V/p$. From the plane wave expansion of the four-component Majorana field given in Eq.(84), it is clear that the $(1+\gamma_5)$ factor in the weak charged current picks up only the coefficients $\alpha^{(-)}_{kn}(p)$. In the relativistic approximation the matrix $\alpha^{(-)}(p)$ is unitary (Eq.(92)), therefore the relativistic approximation for the weak states in matter is

$$|\nu_\alpha(\vec{p},-)\rangle = \sum_{k,n=1,2} U^*_{\alpha k}\alpha^{(-)*}_{kn}(p)|\chi_n(\vec{p},-)\rangle. \tag{97}$$

The mixing matrix between the weak states and the energy eigenstates in matter is given by

$$S^{(-)}(p) = U^*\alpha^{(-)*}(p) = \begin{pmatrix} \cos\varphi^{(-)}(p) & \sin\varphi^{(-)}(p) \\ -\sin\varphi^{(-)}(p) & \cos\varphi^{(-)}(p) \end{pmatrix} \tag{98}$$

where $\varphi^{(-)}(p)$ is the effective mixing angle in matter given by

$$\varphi^{(-)}(p) = \vartheta + \vartheta^{(-)}(p). \tag{99}$$

Thus to the lowest order of relativistic approximation the mixing matrix of the states in matter does not depend on the CP violating phase factor β.

The time evolution equation Eq.(96) of the weak states becomes

$$\left(i\frac{d}{dt} - p - \frac{\Sigma^{(-)}}{4p}\right)\begin{pmatrix} |\nu_e(\vec{p},-;t)\rangle \\ |\nu_\mu(\vec{p},-;t)\rangle \end{pmatrix} =$$

$$= \frac{1}{4p}\begin{pmatrix} -\Delta^{(-)}\cos\left(2\varphi^{(-)}(p)\right) & \Delta^{(-)}\sin\left(2\varphi^{(-)}(p)\right) \\ \Delta^{(-)}\sin\left(2\varphi^{(-)}(p)\right) & \Delta^{(-)}\cos\left(2\varphi^{(-)}(p)\right) \end{pmatrix}\begin{pmatrix} |\nu_e(\vec{p},-;t)\rangle \\ |\nu_\mu(\vec{p},-;t)\rangle \end{pmatrix} \tag{100}$$

where

$$\Sigma^{(-)} = m_1^{(-)2} + m_2^{(-)2}. \tag{101}$$

From Eq.s(87), (89) and (91)

$$\Sigma^{(-)} = \Sigma + 4p\left(V_N + \frac{V_C}{2}\right) \tag{102}$$

$$\Delta^{(-)} \sin\left(2\varphi^{(-)}(p)\right) = \Delta \sin(2\vartheta) \tag{103}$$

$$\Delta^{(-)} \cos\left(2\varphi^{(-)}(p)\right) = \Delta \cos(2\vartheta) - 2pV_C \tag{104}$$

and the time evolution equation Eq.(100) of the weak states can be written as

$$\left(i\frac{d}{dt} - p - \frac{\Sigma}{4p} - V_N - \frac{V_C}{2}\right)\begin{pmatrix} |\nu_e(\vec{p}, -; t)\rangle \\ \\ |\nu_\mu(\vec{p}, -; t)\rangle \end{pmatrix} =$$

$$= \frac{1}{4p}\begin{pmatrix} -\Delta\cos(2\vartheta) + 2pV_C & \Delta\sin(2\vartheta) \\ \\ \Delta\sin(2\vartheta) & \Delta\cos(2\vartheta) - 2pV_C \end{pmatrix}\begin{pmatrix} |\nu_e(\vec{p}, -; t)\rangle \\ \\ |\nu_\mu(\vec{p}, -; t)\rangle \end{pmatrix} \tag{105}$$

This equation is the usual MSW time evolution equation for the neutrino oscillations in matter [13, 14, 15]. Since the MSW time evolution equation does not depend on the CP violating phase β, there is no CP violation effect in the flavour neutrino oscillations in matter.

As in the case of the neutrino flavour oscillations in vacuum, in order to see if the CP violating phases characteristic of Majorana neutrinos are physically observable quantities in some neutrino oscillation experiment, one must consider a lepton number non-conserving process. For example, consider the same process as described in section 4 but now the neutrino is produced, propagate and is detected in a dense medium. The amplitude for the overall process is given by the sum over the index n, that denotes the energy eigenstates, and by the average over the neutrino helicities of the product of the following three amplitudes:

1. The amplitude $A_1^{(n,\pm)}$ for the interaction $\ell_\alpha^+ + n \to \chi_n(\pm) + p$, where the notation $(\pm)$ denotes the helicity of the Majorana neutrino:

$$A_1^{(n,\pm)} = \frac{G_F}{\sqrt{2}} \sum_k \langle \chi_n(\pm)|\overline{\ell_\alpha}(0)\gamma^\lambda(1+\gamma_5)U_{\alpha k}\chi_k(0)|\ell_\alpha^+\rangle \; h_{1\lambda}^{(n)}$$

$$= \frac{G_F}{\sqrt{2}} \sum_k \overline{v_{\ell_\alpha}} \gamma^\lambda (1+\gamma_5) U_{\alpha k} \begin{pmatrix} \mp\, \beta_{kn}^{(\pm)*}(p)\ \omega(\vec{p},\mp) \\ \\ \pm \alpha_{kn}^{(\pm)*}(p)\ \omega(\vec{p},\mp) \end{pmatrix} h_{1\lambda}^{(n)}. \tag{106}$$

2. The amplitude for the energy eigenstate neutrino $\chi_n(\pm)$ to propagate from the initial time $t=0$ to the final time t:

$$\exp\left\{-i\int_0^t E_n^{(\pm)}\ dt\right\} \simeq e^{-ipt} \exp\left\{-i\int_0^t \frac{m_n^{(\pm)2}}{2p}\ dt\right\}. \tag{107}$$

3. The amplitude $A_2^{(n,\pm)}$ for the interaction $\chi_k(\pm) + n \to \ell_\beta^- + p$:

$$A_2^{(n,\pm)} = \frac{G_F}{\sqrt{2}} \sum_j \langle \ell_\beta^- | \overline{\ell_\beta}(0) \gamma^\rho (1+\gamma_5) U_{\beta j} \chi_j(0) | \chi_n(\pm)\rangle\ h_{2\rho}^{(n)}$$

$$= \frac{G_F}{\sqrt{2}} \sum_j \overline{u_{\ell_\beta}} \gamma^\rho (1+\gamma_5) U_{\beta j} \begin{pmatrix} \alpha_{kn}^{(\pm)}(p)\ \omega(\vec{p},\pm) \\ \\ \beta_{kn}^{(\pm)}(p)\ \omega(\vec{p},\pm) \end{pmatrix} h_{2\rho}^{(n)}. \tag{108}$$

Using the properties of the two-component helicity eigenstate spinors given in Eq.(69), it is straightforward to show that the overall amplitude for the lepton number non-conserving process under consideration is proportional to

$$\begin{aligned} A_{\bar\alpha\to\beta}(t) \sim \sum_{n,j,k} U_{\beta j} U_{\alpha k} \Bigg[& \frac{m_k}{p}\ \alpha_{jn}^{(-)}(p) \alpha_{kn}^{(-)*}(p)\ \exp\left\{-i\int_0^t \frac{m_n^{(-)2}}{2p} dt\right\} \times \\ & \times \overline{u_{\ell_\beta}} \gamma^\rho \left(1 + \frac{\vec{\gamma}\cdot\vec{p}}{p}\gamma_0\right)(1+\gamma_5)\gamma^\lambda u_{\ell_\alpha} + \\ & + \frac{m_j}{p} \beta_{jn}^{(+)}(p) \beta_{kn}^{(+)*}(p) \exp\left\{-i\int_0^t \frac{m_n^{(+)2}}{2p} dt\right\} \times \\ & \times \overline{u_{\ell_\beta}} \gamma^\rho \left(1 - \frac{\vec{\gamma}\cdot\vec{p}}{p}\gamma_0\right)(1+\gamma_5)\gamma^\lambda u_{\ell_\alpha} \Bigg] h_{1\lambda}^{(n)} h_{2\rho}^{(n)}. \end{aligned} \tag{109}$$

Unfortunately, the expression given above has no clear physical usefulness, since the charged lepton part of the amplitude cannot be factorized out. In any case, as in vacuum, the probabilities $P_{\bar\alpha\to\beta}(t) = |A_{\bar\alpha\to\beta}(t)|^2$ depend explicitly on the CP violating phase, but they are strongly suppressed for relativistic neutrinos by a factor of order m^2/p^2.

8 Conclusions

We have demonstrated that the CP violating phase cannot appear in the flavour oscillation probability of two generations of Majorana neutrinos in matter, as well as in vacuum. When the neutrinos propagate in a medium, their energy eigenvalues depend explicitly upon the CP violating phase characteristic of Majorana neutrinos, implying that the medium can induce CP violation effects. However the CP violation effects disappear in the flavour oscillation probability. More specifically, we have derived the MSW equation by defining the weak neutrino states, namely the states that describe the neutrinos produced or detected in a weak interaction process, as a superposition of energy eigenstates in the lowest order of the relativistic approximation. To this order we have shown that there is no CP violation effect, in spite of the presence of the CP violating phase in the mixing matrix of the fields in vacuum and in the quantized neutrino fields in matter. On the other hand, the CP violating phase does appear in the lepton number non-conserving flavour oscillations that are characteristic of the Majorana neutrinos, both in vacuum and in matter, but the oscillation probabilities are always suppressed by a factor of order m^2/p^2 that forsake their experimental usefulness. Therefore, the only promising process in which the CP violation phase may play an important role is the neutrinoless double beta decay. Once the neutrino properties, such as the masses and the mixing angles, are determined independently from other processes, the future observation of the neutrinoless double beta decay can reveal the nature of the CP violation in the lepton sector.

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