Lecture 4:
Some approximate Riemann solvers
for the Euler equations

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Recalling the Godunov Scheme

\[ \partial_t Q + \partial_x F(Q) = 0 \]

\[ Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} \left[ F_{i+1/2} - F_{i-1/2} \right] \]

To compute \( F_{i+1/2}^{GOD} \) we must solve the Riemann problem

\[ \partial_t Q + \partial_x F(Q) = 0 \]

\[ Q(x,0) = \begin{cases} Q_i^n, & x < x_{i+1/2} \\ Q_{i+1}^n, & x > x_{i+1/2} \end{cases} \rightarrow Q_{i+1/2}(x/t) \]

\[ F_{i+1/2} = \frac{1}{\Delta t} \int_0^{\Delta t} F(Q_{i+1/2}(0)) \, dt = F(Q_{i+1/2}(0)) \]
3D case in normal direction

\[ Q_t + F(Q)_x = 0 \]

\[ Q = \begin{bmatrix}
\rho \\
\rho u \\
\rho v \\
\rho w \\
E
\end{bmatrix} \quad F(Q) = \begin{bmatrix}
\rho u \\
\rho u^2 + p \\
\rho uv \\
\rho uw \\
u(E + p)
\end{bmatrix} \]

*Fig. 4.18. Structure of the solution of the Riemann problem for the split three-dimensional case*
In general, for the 1D Euler equations, there are 10 possible wave configurations to consider in the solution sampling. See Fig. below.
Approximate Riemann solvers.
The classics:

LINEAR SOLVERS:
• Godunov (1961)
• Roe (1981)

NON-LINEAR SOLVERS:
• Osher-Solomon (1982)
• Harten, Lax and van Leer (1983)

These contain the fundamental ideas but are not necessarily the best Riemann solvers. Beware!
The Harten-Lax-van Leer approach (HLL) 1983
Assume we know estimates for the smallest and largest signal speeds arising from the solution of the Riemann problem:

Note: all intermediate waves in the structure of the solution are ignored.
Application of the integral form of the conservation laws gives the HLL state:

$$\frac{1}{T(S_R-S_L)} \int_{T_S}^{T_R} Q(x, T)dx = \frac{S_R U_R - S_L U_L + F_L - F_R}{S_R - S_L} = Q^{hll}$$

This is the HLL average state. This is not what is used for flux evaluation.

It is easy to show that use of the integral form of the conservation laws gives the HLL flux

$$F^{hll} = \frac{S_R F_L - S_L F_R + S_R S_L (Q_R - Q_L)}{S_R - S_L}$$
The HLL intercell numerical flux is:

\[
F_{i+1/2}^{hll} = \begin{cases} 
  F_L, & 0 \leq S_L \\
  \frac{S_R F_L - S_L F_R + S_R S_L (Q_R - Q_L)}{S_R - S_L}, & S_L \leq 0 \leq S_R \\
  F_R, & 0 \geq S_R 
\end{cases}
\]

Pending task: find estimates for wave speeds (later)

- This Riemann solver is non-linear
- Wave speed estimates are still needed, for which knowledge of the solution is required in advance. See Davis (1988) and Einfeldt (1988)
- Both Davis and Einfeldt showed their HLL schemes to possess some very good properties
A weak feature of HLL:

absence of intermediate waves:

In particular:

• Entropy waves
• Slip surfaces
• Material interfaces
• Vortical flows
• Ignition fronts
• Shear layers
• Contact discontinuities
HLL (left) HLLC (right)

Fig. 10.20. Godunov’s method with HLL (left) and HLLC (right) Riemann solvers applied to Tests 6 and 7. Numerical (symbol) and exact (line) solutions are compared at time 2.0
Futher reading on the HLL Riemann solver


..........................and references their in...
The HLLC Riemann solver
T et al. (1992, 1994)

A quick search with google gave me:

HLLC:  Healesville Living and Learning Centre
HLLC:  Happy Land Learning Center
HLLC:  House of Lords Liaison Committee
HLLC:  Home Loan Learning Center
HLLC:  Harten, Lax, van Leer and (the missing) Contact
The HLLC solver (Toro et al. 1992, 1994) is a modification of the HLL Riemann solve

* C stands for CONTACT *

The contact wave is included in the structure of the solution of the Riemann problem

Now the Star Region has two sub-regions (for a 3 by 3 system)

Further developments on HLLC:
Toro and Chakraborty, 1994
Batten et al. 1997a, 1997b
The HLLC Riemann Solver (cont....)

\[ F_{i+1/2}^{hllc} = \begin{cases} 
F_L, & 0 \leq S_L \\
F_{*L}, & S_L \leq 0 \leq S_* \\
F_{*R}, & S_* \leq 0 \leq S_R \\
F_R, & 0 \geq S_R 
\end{cases} \]

\[ F_{*L} = F_L + S_L (Q_{*L} - Q_L), \quad F_{*R} = F_R + S_R (Q_{*R} - Q_R) \]
We have 4 unknown vectors:

\[ Q^*_L, \quad Q^*_R, \quad F^*_L, \quad F^*_R \]

First solve for the states:

\[ Q^*_L, \quad Q^*_R \]

Then solve for the fluxes:

\[ F^*_L, \quad F^*_R \]

We assume the following conditions in the star region:

\[
\begin{align*}
    p^*_L &= p^*_R = p^*_* \\
    u^*_L &= u^*_R = u^*_* = S^*_* \\
    v^*_L &= v^*_L \\
    v^*_R &= v^*_R \\
    w^*_L &= w^*_L \\
    w^*_R &= w^*_R
\end{align*}
\]

These conditions are satisfied by the exact solution. See Toro 1999 (Springer).
Then we can write

\[ S_L Q^*_L - F^*_L = M_L (Q_L, Q_R) \]
\[ S_R Q^*_R - F^*_R = M_R (Q_L, Q_R) \]

where the right hand sides are known functions.

Algebraic manipulations give the solution for the unknown states, from which the sought flux vectors follow.
\[ F_{*L} = F_L + S_L (Q_{*L} - Q_L), \quad F_{*R} = F_R + S_R (Q_{*R} - Q_R) \]

\[ Q_{*K} = \rho_K \left( \frac{S_K - u_K}{S_K - S_*} \right) \left[ \begin{array}{c} 1 \\ S_* \\ \nu_K \\ W_K \\ \frac{E_K}{\rho_K} + (S_* - u_K) \left[ S_* + \frac{p_K}{\rho_K (S_K - u_K)} \right] \end{array} \right] \quad K = L, R \]
The 3D multi-component case

\[ Q_t + F(Q)_x = 0 \]

\[ Q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \\ \rho \phi_1 \\ \rho \phi_2 \\ \vdots \\ \rho \phi_m \end{bmatrix}, \quad F(Q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ u(E + p) \\ u \rho \phi_1 \\ \vdots \\ u \rho \phi_m \end{bmatrix} \]

Obtain eigenvalue \( u \) of multiplicity \( m+3 \) and the HLLC procedure goes through.
Wave-Speed Estimates for HLLC

We need estimates $S_L, S_*, S_R$

Find estimates for $p_*, u_*$

$$S_L = u_L - a_L q_L, \quad S_* = u_*, \quad S_R = u_R + a_R q_R$$

$$q_K = \begin{cases} 
1 & \text{rarefaction} \\
\sqrt{1 + \frac{\gamma + 1}{2\gamma} \left( \frac{p_*}{p_K} - 1 \right)} & \text{shock}
\end{cases} \quad p_* \leq p_K$$

$$p_* > p_K$$
• Linearize the non-conservative system:

\[ W_t + A(W)W_x = 0 \]

by ‘freezing’ coefficient matrix at a state \( \hat{W} \)

• Standard linear theory gives the explicit solution:

\[
\begin{align*}
p_* &= \frac{1}{2} (p_L + p_R) + \frac{1}{2} (u_L - u_R)(\hat{\rho}\hat{a}) \\
u_* &= \frac{1}{2} (u_L + u_R) + \frac{1}{2} (p_L - p_R)/(\hat{\rho}\hat{a})
\end{align*}
\]
HLLC--summary

\[ p_\ast = \frac{1}{2} (p_L + p_R) + \frac{1}{2} (u_L - u_R)(\hat{\rho} \hat{a}) \]
\[ u_\ast = \frac{1}{2} (u_L + u_R) + \frac{1}{2} (p_L - p_R)/(\hat{\rho} \hat{a}) \]
\[ \hat{\rho} = \frac{1}{2} (\rho_L + \rho_R) \quad \hat{a} = \frac{1}{2} (a_L + a_R) \]

\[ S_L = u_L - a_L q_L, \quad S_\ast = u_\ast, \quad S_R = u_R + a_R q_R \]

\[ F_{\text{hllc}}^{i+1/2} = \begin{cases} 
F_L, & 0 \leq S_L \\
F_\ast L, & S_L \leq 0 \leq S_\ast \\
F_\ast R, & S_\ast \leq 0 \leq S_R \\
F_R, & 0 \geq S_R 
\end{cases} \]

\[ F_{\ast L} = F_L + S_L (Q_{\ast L} - Q_L), \quad F_{\ast R} = F_R + S_R (Q_{\ast R} - Q_R) \]

\[ Q_{\ast K} = \rho_K \left( \frac{S_K - u_K}{S_K - S_{\ast}} \right) \]

\[ \begin{bmatrix}
1 \\
v_K \\
w_K \\
\frac{E_K}{\rho_K} + (S_\ast - u_K) \left[ S_\ast + \frac{p_K}{\rho_K (S_K - u_K)} \right]
\end{bmatrix} \quad K = L, R \]
HLL versus HLLC

Fig. 10.20. Godunov’s method with HLL (left) and HLLC (right) Riemann solvers applied to Tests 6 and 7. Numerical (symbol) and exact (line) solutions are compared at time 2.0
Extensions and Applications of HLLC

• Shallow water equations (Toro, 2001)
• MHD (K F Gurski, SIAM J Sc Comput., 2004)
• MHD (Shengta Li, JCP)
• Relativistic MHD: Mignone, Massaglia and Bodo
• High-order extensions via:
  the WAF method (Toro, 1989)
  ADER method (Toro et al. 2001 and others)
  WENO methods (Titarev and Toro)
  Discontinuous Galerkin Finite Element Methods (van der Vegt, 2002)

• 2D multiphase flows (Toro, 1992)
• Implicit version for compressible turbulent flows (Batten et al. 1997)
• Multiphase, multi-dimensional flows (Toro 1992, Saurel, 2002)
• …. and many more, including packages and commercial software.
  http://vulcan-cfd.larc.nasa.gov/index.html
Futher reading on the HLLC Riemann solver


(Also as Cranfield University Technical Report, 1992)


The Rusanov Riemann solver (1961) and the Lax-Friedrichs flux (1960)
Rusanov’s flux (1961)

Consider the HLL flux

\[ F^{hll} = \frac{S_R F_L - S_L F_R + S_R S_L (Q_R - Q_L)}{S_R - S_L} \]

Two wave speed estimates are needed: \( S_L \); \( S_R \)

Assume a single wave speed estimate: \( S_+ > 0 \)

Define a second speed: \( S_L = -S_+ \); \( S_R = +S_+ \)

Substitution into HLL flux gives the Rusanov flux

\[ F^{\text{Rusavov}} = \frac{1}{2} (F_L + F_R) - \frac{S_+}{2} (Q_R - Q_L) \]

This flux is sometimes called (wrongly in my view) the Local Lax-Friedrichs or simply the Lax-Friedrichs flux.
Note that if in the Rusanov flux

$$F^\text{Rusavov} = \frac{1}{2} (F_L + F_R) - \frac{S_+}{2} (Q_R - Q_L)$$

$$S_+ = \frac{\Delta x}{\Delta t}$$

we then reproduce the Lax-Friedrichs flux.

$$F^{\text{Lax-F}} = \frac{1}{2} (F_L + F_R) - \frac{1}{2} \frac{\Delta x}{\Delta t} (Q_R - Q_L)$$

In this sense the (centred) Lax-Friedrichs flux can be seen as un upwind flux (the limiting case).
The MUSTA approximate “Riemann solver”
2003
The MUSTA approach

The MUSTA approach (Toro, 2003) is an attempt to regain upwind information but without solving the Riemann problem in the classical sense.

We look for *upwind* schemes that are simple and directly applicable to very complicated problems.

A degree of success has been achieved but the work is not complete.
Schemes associated to the FORCE flux: a motivating discussion

\[ F_{i+1/2}^{(\omega)} = \omega F_{i+1/2}^{\text{LW}} + (1 - \omega) F_{i+1/2}^{\text{LF}} , \quad 0 < \omega < 1 \]

Special cases:

- \( \omega = 0 \) (Lax - Friedrichs)
- \( \omega = 1 \) (Lax - Wendroff)
- \( \omega = 1/2 \) (FORCE)
- \( \omega = \frac{1}{1 + c} \) (GFORCE)
Purpose of MUSTA:

To recover the lost corner by opening the Riemann fan, but without solving the Riemann problem in the classical sense.
Idea: open the fun numerically

Objection: cost, due to
• number of cells
• and number of stages k
MUSTA: multi-stage predictor-corrector approach

Upwinding sneaks up on us!
Monotonicity is the killer

This is MUSTA with one stage and 2 local cells using FORCE as predictor and corrector (Toro 2003)

Recent analysis (Titarev and Toro, 2006) shows that as the number of stages is increased, while keeping the number of cells constant, worsens the situation
Two classes of MUSTA schemes:

- Use FORCE (or GFORCE) as Predictor (many stages on a sufficiently large mesh) and Corrector (final stage) stages

Simple and general but (may be) costly
Research still in progress

More details in:
Two classes of MUSTA schemes (cont...)

- The EVILIN variant

- Perform a single predictor stage using FORCE or GFORCE

- Perform a linearization on evolved states and solve simple linearized Riemann problem
  
  - The resulting Riemann solver is complete
  
  - It is simple, as long as we can solve a linear system
  
  - But one needs the eigenstructure of the system
  
  - It is entropy satisfying

Further details on EVILIN variant in:

Toro EF. Riemann solvers with evolved initial conditions.
The MUSTA-1 scheme

Predictor step

\[ \hat{Q}_i^n = Q_i^n - \frac{\Delta d}{\Delta \tau} \left[ F^{gforce} (Q_i^n, Q_{i+1}^n) - F(Q_i^n) \right] \]
\[ \hat{Q}_{i+1}^n = Q_{i+1}^n - \frac{\Delta d}{\Delta \tau} \left[ F(Q_{i+1}^n) - F^{gforce} (Q_i^n, Q_{i+1}^n) \right] \]

Corrector step

\[ F^{GF}_{i+1/2} = \begin{cases}  
F(\hat{W}_{1/2}(0)) & \\
F^{gforce} (\hat{Q}_i^n, \hat{Q}_{i+1}^n) & 
\end{cases} \]
The EVILIN variant for the 3D Euler equations

Corrector step: solve linear Riemann problem

\[
\begin{align*}
\partial_t W + B \partial_x W &= 0 \\
W(x,0) &= \begin{cases} 
W_L = W_L(\hat{Q}_i^n), & x < 0 \\
W_R = W_R(\hat{Q}_{i+1}^n), & x > 0 
\end{cases} \\
\hat{B} &= B(\frac{1}{2}(\hat{Q}_i^n + \hat{Q}_{i+1}^n)) \\
\end{align*}
\]

\[
\alpha_1 R^{(1)} + \alpha_2 R^{(2)} + \alpha_3 R^{(3)} + \alpha_4 R^{(4)} + \alpha_5 R^{(5)} = \Delta \equiv W_R(\hat{Q}_{i+1}^n) - W_L(\hat{Q}_i^n)
\]

\[
\begin{align*}
\alpha_1 &= \frac{\Delta p - \Delta u\hat{\rho}a}{2\hat{\rho}a^2} \\
\alpha_2 &= \frac{\Delta \rho\hat{a}^2 - \Delta p}{a^2} \\
\alpha_3 &= \Delta v \\
\alpha_4 &= \Delta w \\
\alpha_5 &= \frac{\Delta p + \Delta u\hat{\rho}a}{2\hat{\rho}a^2} \\
\Delta q &= \hat{q}_R - \hat{q}_L
\end{align*}
\]

\[
\hat{W}_{1/2}(0) = W_L + \sum_{\hat{\lambda}^{(i)} < 0} \gamma_i R^{(i)} \quad \text{or} \quad \hat{W}_{1/2}(0) = W_R - \sum_{\hat{\lambda}^{(i)} > 0} \gamma_i R^{(i)}
\]

\[
F_{i+1/2}^{\text{God}} = F(\hat{W}_{1/2}(0))
\]
Some test problems for EVILIN applied to the Euler equations.
The 123 test (low density)
Sonic flow test
Isolated stationary contact
The Woodward and Colella blast wave test
The Woodward and Colella blast wave test
Mach reflection (2D)
Mach reflection (2D)
Further reading on the MUSTA “Riemann solver”


Hierarchy of Riemann solvers:

**Complete Riemann solvers:** Contain in their structure the full set of waves present in the exact solution of the Riemann problem

- Exact Riemann solver
- Roe’s Riemann solver
- Osher’s Riemann solver
- HLLC Riemann solver (for Euler+)
- EVILIN

**Incomplete Riemann solvers:** do not contain in their structure the full set of waves present in the exact solution of the Riemann problem

- HLL Riemann solver
- FVS fluxes
- Rusanov
- MUSTA
- GFORCE

**Centred (symmetric):** do not explicitly contain any information on the wave structure of the exact solution of the Riemann problem

- Lax-Friedrichs, FORCE
Concluding remarks on classical Riemann solvers

- Discussion restricted to the classical case (piece-wise constant data)
- The discussion has not been comprehensive
- The ultimate classical solver not yet available: non-linear and complete
- Classical Riemann solvers give the numerical flux for the first-order Godunov upwind method
- High-order Godunov-type methods can be constructed using classical Riemann solvers (low order)

The high-order Riemann problem

It is the Cauchy (IVP) with spatially high-order data, such as piece-wise polynomials

The solution at the interface is a high-degree polynomial in time